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Mathematics and Intellectual Abilities*

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Introduction

Although your present speaker is not directly engaged in the problems of curriculum, he has a very strong belief in the importance of mathematics and exact sciences in modern education. In passing, he merely wishes to note that we are living in a modern world requiring exactness, precision, split-second decisions, balance sheets that balance, and bridges that do not collapse. All of these considerations are related in practical ways to present-day education.

On the other side, we have children in various stages of mental development entering and passing through our schools. Many are average, some are bright, others are dull. Some have one equipment of mental abilities, others have something entirely different. It is the speaker's purpose to relate intellectual abilities to problems of mathematics instruction.

Number Backgrounds

For minds that wish to be very concrete, mathematics and number concepts are highly satisfying. Ideas of numbers in some practical form very early come to the attention of children. The toddling baby soon learns to distinguish between one and two pieces of candy. Small urchins quickly appreciate the difference between 5¢ and 6¢ in what may be purchased at the neighborhood store. It is evident from a common sense point of view that children have some pretty definite, although simple, number concepts well established long before they enter the kindergarten or the first grade. In this respect mathematics is unique among the school subjects. Very few children know how to spell at that age, to read, to write, or to comprehend facts of our historical backgrounds. It is a profound mystery to me why this field of practical arithmetic information is not made the starting point for school activities more generally than is found in the present-day school curriculums. In its place we now labor with the very complex process of learning how to read in the early days of the first grade. It has been shown through many elaborate experiments that, unless the child's mental maturity is about 6 years, he finds great difficulty with reading. Although everyone knows that there are thousands of children in the first grade not yet ready, we are slow to change our policies in these regards. At this point I wish to remark

* An address given before the National Council of Teachers of Mathematics at Detroit, Michigan on June 30, 1937.

that we are not advocating an immediate beginning of abstract mathematics or material which is now taught in the second or third grades, but preliminary to such material the number and form concepts of children could be much more effectively utilized in and about the kindergarten and first grade level. Several years ago Dr. Buckingham reported on a survey of children's number knowledge before they entered the first grade. The great majority of children could count at least as high as 20 understandingly and many other of the simpler facts were already known. It seems high time that these general facts and conditions should be placed into more effective curriculum procedures in the lower elementary grades.

Advanced Mathematics

In the fields at the high school and college levels there are even greater opportunities for the utilization of mathematics. These fields of exact sciences are not closed books since there are probably as yet many new mathematical theorems to be developed, new realms of logic to be explored. In contrast to this point of view, the study of foreign languages, such as Classical Latin, is limited to the present known original works in that language. The main facts about the civilization of ancient Greece are also quite well defined. But who can say that all the formulas of mathematics have been discovered? These exact sciences, therefore, are a fertile field of exploration which should challenge the more intelligent of our students in high school and college. If the challenge is not effective, the fault must partly lie within our own poor salesmanship of the wares closest to our own interest. We must now turn to other considerations with respect to children and their learning of mathematics.

Mental Maturity

It has already been noted that the simplest kind of number concepts develop at a very early age, before the time of school entrance. Other concepts appear at later stages of development. Washburn did certain constructive research along these lines several years ago. He set a criterion of 80% correct retention in three out of four pupils. On this standard for sums of 10 and under a mental age of 6 years 5 months; easier subtraction 7 years; multiplication facts 8 years 4 months; simple long division 10 years 9 months, case I percentage 12 years 4 months. It is evident from Washburn's findings that we should more accurately grade our mathematical material with respect to mental maturity.

For the higher mental levels there seems to be almost no upper limit of the possibilities of mathematical concepts. This is particularly true if the early training and development have been logically and skillfully executed. In the carrying out of such projects it is desirable that the psychologist and the mathematician work hand in hand since the problem has mutual interests of both. For the mathematician it should appeal as a problem of practical advancement in the fields of mathematics. To the psychologist it is a field of interest, investigating particularly the higher mental processes their instructional opportunities and outlets.

Qualitative Mental Characteristics

In the previous section we have presented the consideration of general mental age as a factor in the knowledge and mastery of mathematics. In the present section we wish to continue this topic further along the line of brightness and dullness, which interprets the I.Q. rather than the mental age.

The concept of bright, average and dull children has become a common byword in the fields of education. In the past 20 years it has become more specific since the use of psychological tests with I.Q.'s or with the letter ratings, which correspond to them. Approximately one-fourth of the population have I.Q.'s of 90 or less, and one-fourth 110 or higher.

Practically, we will now address ourselves to these groups. Even though we were to consider older dull children with I.Q.'s of 85 and the same mental age as younger bright children with I.Q.'s of 115, the ways in which these two groups respond to learning situations, including mathematics, are worthy of our careful consideration. If anyone has taught extremely gifted or extremely backward children or has had extensive experience in the administration of psychological tests, certain characteristics of response are obvious and rather well known. However, it seems desirable to review a few of these at this point. This review is also to be linked with practical problems of instruction.

1. *Learning bonds.* The child with a low I.Q. tends to learn by simple and direct methods, mainly of rote repetition. Even at this level his learning tends to be faulty and inaccurate. It is best executed by a certain amount of rote learning so arranged as not to bring about mental fatigue and lack of interest. Certain number concepts may be acquired through this rote mastery. On the other hand, bright children tend to learn by a many-track mental plan of learning bonds. They learn in many different ways rather than by one simple process. We characterize this phenomenon by saying that they have rich powers of associative learning. When they wish to recall some fact or to establish some bond, they do it by invoking these many avenues of association rather than the recall of one simple bond executed many times. In other words, learning to associate a person's name with his appearance upon the part of bright children consists of associating the name with the city, the name with someone else who looks similar, with some article of dress, with some mannerism, and other common leads. On account of searching for many avenues of learning, the bright child at times seems exasperatingly slow in his reactions. As a matter of fact, his mental actions are proceeding at a very rapid rate but are not obvious since they are often unexpressed.

In interpreting this difference between brightness and dullness, one must not be too literally emphatic upon such trends. If so, one would merely make all learning of dull children a grinding mill for rote learning and would wait patiently for undue periods for the responses of the bright. Even among those who are bright the rote processes should not be entirely neglected. They still must serve, to a certain extent, as a foundation for any effective mental processes. It would also be a fine art in learning for the dull if they could expand their learning from simple situations to at least two choices. If the two can be made to function, the learning will be many times more efficient than one only. Although psychologists and teachers are aware of these trends in simple versus associative learning, there have been practically no dissertations devoted to further exploration on this topic. Its possibilities have really never been fathomed. The psychologists' assertion that there are such differences has been questioned, although long-time observation of many children points pretty definitely toward the hypothesis here presented.

2. *Concrete versus abstract learning.* The mentally slow child tends to think in terms of the concrete and specific situation. The immediate problem which he is doing now does not necessarily lead to understanding other problems. The brighter children tend to understand more generalized relationships of an abstract nature. Although it is desirable to proceed from the concrete to the abstract, even with them, the major emphasis should be laid upon the abstract and generalized. No one probably knows just how much of the concrete is necessary for a bright child before the abstract can be made to function effectively. If we could really get this concept operating as it should, the fields of mathematics at the high school and college level could take on

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a new lease on life. They would appeal to a much larger number of mentally superior students.

Whatever ray of hope there is supposed to be in the so-called transfer of training from one subject to another, we might hope to find some ability to generalize in mathematics carried over into other subjects. It is desirable to note that these characteristics are trends rather than mutually exclusive concepts. In the case of the mentally gifted with extremely high I.Q.'s there seems to be almost no limitation to the principle of abstractions. One sincerely hopes that eventually we may go to that limit in finding out more about this important topic.

3. *Theory versus practice.* There seems to be a characteristic difference between children of high I.Q. and low I.Q. with regard to the interest and emphasis which they put upon practice as contrasted to theory. In other words, dull children seem to be much more interested in doing the practical allied work of any course rather than understanding the theories and principles with regard to it. This interest and emphasis upon practice does not necessarily guarantee a performance in these fields superior to that of the more gifted children. In fact, it is usually quite the opposite since the dull do not understand the necessary theory which accompanies it nor can they actually carry out even physical manipulations as well as the mentally superior. A somewhat superficial emphasis upon the physical side of all courses is too often incorrectly interpreted to be superior performance. In teaching the dull, we might well capitalize this natural interest in the practical but along with it try to develop the theoretical side in its simplest forms as much as the traffic will stand.

In like manner bright children tend to give more attention to theory and care relatively less about the concrete and practical applications. Whenever courses are offered in which the practical is emphasized without due respect to the theory, bright children lose their interest in those courses. It is a fine art to keep a proper balance between these factors and to understand their relationships at all I.Q. levels.

4. *Units of Work and Coordination.* From time to time various theories and isms are ardently advocated as the cure-all to our problems of instruction. It is assumed that they will work effectively at all levels of instruction. A case in point is the unit system of teaching. While a controversy may be started over the length of units, in brief the brighter the child the longer the unit may become, the duller the child the shorter and more specific it needs to be. In these shorter units the dull child must also have a skillful review of previous units, otherwise his short memory span tricks him into losing the significance of the present unit.

In this same connection it may be mentioned that dull children see comparatively little relationship between any two courses of the curriculum. Children of higher I.Q. may easily be trained in quite the opposite point of view. Within a departmentalized school teachers of different departments compliment each other for cooperation when thinking about bright children and condemn each other when considering problems of the dull. It would seem a bit more logical to study and understand this problem with respect to the carry-over possible within the children themselves. Bright children seem to do this rather naturally and to find practical relationships of numbers and mathematics to other fields of instruction. The dull child must be carefully built up to some specific relationships by the academic or homeroom teacher. The special teacher, hoping to make applications of mathematical information, must continue where the other teacher left off and definitely point out these relationships to dull pupils.

5. *Other differences.* It has many times been argued that the mental training from mathematics and some other sub-

jects has an important carry-over effect to other subjects of the curriculum and to problems of life. This question is, undoubtedly, related to the qualifications going with different levels of brightness. It has been shown by various studies that children do excel in certain subjects after they have excelled in others. Whether this was due to the experience and carry-over or to the natural abilities of children is a matter for further experimentation. This problem is related to that psychological trait known as reasoning. Reasoning is the ability to do, mentally, processes and operations which otherwise would have to be done by the process of trial and error. Dull children seem to be unusually lacking in this ability to visualize and to create the solution of problems. On the other hand, bright children can perform brilliantly in these regards. We usually overlook some golden opportunities of reasoning on the part of bright children. This whole question of reasoning of the so-called transfer of training and allied topics should be reviewed again in the light of our knowledge of the mental abilities and characteristics of children.

Somewhat indirectly related from the standpoint of mathematics but practically related from the standpoint of teaching children are other characteristics of the bright and dull children. Bright children generally have high social intelligence and interests outside of themselves. Dull children tend to be self-centered and to take issues chiefly on a personal basis. It should be one of the duties of all teachers, including those of mathematics, to devote their energies to a better development of the social intelligence of the mentally slow in their classes.

Children with lower I.Q.'s tend to be more the victim of emotional and developmental traits and moods than are the bright. On the other hand, the bright tend to allow their mental processes to control their emotions. The teacher of the bright who makes error of appealing to emotional side of children at the expense of their better intellectual judgment soon loses control over them. The teacher of the dull who continually tends to reason at the exclusion of their feelings and emotions does not find such pupils and their traits very congenial.

Specialized Mental Traits

Knowledge and ability with numbers have always been important aspects of intelligence. Some form of number knowledge is usually found in tests of intelligence themselves. The Army Alpha Intelligence Test had one page of arithmetic problems, and another of arithmetic number series. Many group tests today have one or more pages with certain number abilities and the Stanford-Binet Test also contains some of them. Dr. L. L. Thurstone has recently attempted to isolate the traits which are in intelligence by statistical methods. A generalized number ability proved to be one of several traits composing intelligence.

In some experiments carried out here in Detroit your speaker has studied the special mental abilities and disabilities of children with respect to success or failure in subjects such as arithmetic in the elementary school. The most extreme cases of strong abilities and of weak abilities were contrasted with respect to certain of these psychological traits. Although only a few children were involved in each group, marked tendencies came to our attention. Children who are very good in mathematics pass a high test in the free association of the Stanford-Binet Test above their general mental age. Children with low mathematical abilities pass low on their free association tests. In other words, ability to associate words rapidly is apparently related to ability in numbers. This result verifies our belief that numbers and number concepts intelligently applied make demands upon powers of association in the human mind. We may have emphasized too much rote memory and learning of facts rather than

the associations which should be built up around number concepts.

Another marked relationship is between visual intelligence and numbers. This is true of a visual rote memory for objects or pictures of objects and also for situations requiring some visual imagery. In our psychological diagnoses of adults those who are weak in numbers almost universally make a poor showing on any psychological test bearing the label of visual imagery and imagination. Illustrations of such tests are the ability to assemble the parts of a disarranged picture whose parts are numbered, the ability to find one's way through a visual maze test, or to solve a block design test with paper and pencil. These general relationships suggest that from the very start children must be able to visualize four things when we mention the number 4. They must eventually be able to get a fairly definite picture of five rows with five each to understand the concept of 25.

We have also discovered that number ability is related to orientation or knowledge of one's environment. The child who lives a sheltered life with his childish needs cared for by others finds little occasion to learn numbers or other subjects in a practical manner. Incidentally, orientation is also related to inadequate handwriting on the part of young children.

These suggestions are only sketchy landmarks in what should be a very promising field of research. They have been mentioned here not as final conclusions but merely as trends brought to light by the data which we have had available. It would seem practical to have leaders and teachers in the field of mathematics joining to urge psychological and educational clinics at centers of learning to discover the real elements which contribute to success or failure in a program of mathematics. Incidentally, there are similar needs in all of their departments of curriculum and instruction.

Conclusion

All of the subjects of the curriculum stand in need of constant study and revision. The needs extend to the aims and purposes as well as to the methods of instruction. All of these phases need interpretation with respect to the potential learning abilities of the child as represented by his I.Q.

Mathematics is an exact science which offers possibilities of exact and scientific analysis with respect to learning processes. If such projects can be carried through in this field at an early date, it may serve as a leader and stimulus to other subjects not so exacting and specific. While the speaker in no sense believes that the I.Q. or intelligence should be the sole determiner of learning, he believes that the relationships of intelligence to these fields have not been utilized to their fullest possibilities.

Gestalt Psychology and Mathematical Insight*

By George W. Hartmann
Teachers College, Columbia University

This paper is presented as an expression of belief and hope that some of the newer formulations in psychological theory will bring about a democratization of mathematical competence somewhat akin to the astonishing elevations of performance often obtained in the field of reading. The symbolism and thought processes involved in using conventional language are not essentially different from those employed in the more universal terminology and operations of mathematics. If reading ability is now better and more widespread among Americans than at any previous time in our history, there is no reason why a better comprehension of numbers, figures, and conceptual relations should not make quantitative inferences as common as simple words and sentences. Although it is probably true that algebra and geometry *as ordinarily taught* are unable to become more influential because of the present intellectual and motivational limitations of the average learner, I should like to maintain the position that by taking more adequate advantage of the principles of mental development, our teachers could make the special modes of thought of the mathematician a part of the daily routine of our citizenry. The deserved success of the Bell and Hogben volumes reveals what some of these possibilities are.

Most persons will probably approve of this end and accept the claim just made, but may maintain that the existing crisis in secondary-school mathematics—a situation which affects more than the jobs of certain teachers of specialized subject-matter—is a consequence of altered social conditions and changed educational attitudes, and that a better methodology and modernized psychology can do little more than make possible a graceful retreat to a more humble role. The essential problem, these folks say, is a curricular one. True enough; but it would be a mistake to assume that there are no psychological foundations to a curriculum, mathematical or otherwise. The human values in both content and procedure are too intimately allied to permit that. It is a common error which holds that an educational discipline becomes "progressive," i.e., modern and enlightened, by skillfully and effectively teaching that which should not be taught and not teaching that which should. The *How* and the *Why* of instruction are organically related and a truly satisfactory solution for one will tend simultaneously to solve the other.

Insofar as any distinct psychological system has been adopted by mathematics teachers, their training appears to have led them to favor the connectionist or "bond" theory of learning, although the frank hostility of this position to the claims of the formal disciplinarians has led many of them to cling desperately to the long-discredited notion of separate mental "faculties." This occurred, one may suppose, not because mathematicians are any more conservative than other pedagogues, but because such an outlook supported their claims to a preferred status in the conventional course of study. In the last decade, however, a small but growing group has found a more satisfactory foundation for its practice in the tenets of the Gestalt brand of psychology —a theory to which mathematicians are perhaps temperamentally congenial because it literally outlines a subtle "geometry of the mind." What are some of the considerations upon which this advanced (and advancing!) viewpoint rests?

* Part of an address delivered on April 10, 1937, before Section 19 (Mathematics) of the New York Society for the Experimental Study of Education.

In my judgment, there are three propositions which are basic to that type of theorizing which goes by the name of Gestalt:

1. *All experience or mental life implies a differentiation of the sensory or perceptual field to which the organism can respond into some kind of figure-ground pattern.* In other words, there must be heterogeneity of stimulation before any psychological process can occur. If we were affected by nothing but undifferentiated homogeneous energy, e.g., a single flat level of grey in vision or an unvarying tonal mass in hearing, the very conditions for consciousness itself would probably be absent. Difference produces phenomena. From this standpoint, variety is more than the spice of life—it is a prerequisite of life itself.

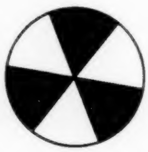


Fig. 1. Note how the black and white propellers are alternately seen.

This dualism of figure and ground is an inescapable feature of all perception, but it is a *functional* and not a structural antithesis. The figure is simply that feature of the situation to which primary attention is given *at the moment*—the ground, although essential to the emergence of the figure, has a secondary role in terms of the focalization of the organism's interest. In Fig. 1 (which is typical of all "reversible" patterns), the black and the white areas alternate in dominating the reader's field; when the black region is figure, the white is ground, and vice versa. Most of the patterns we encounter are far more stable than this, although all can be reorganized subjectively and made to fluctuate to some extent. Thus, the interlinear white space on this page, which is normally "unnoticed" ground even though it is absolutely essential to the reading act, can—with some effort—undergo a transformation and acquire temporarily the status of "figure."

2. *The course of mental development is from a broad, vague and indefinite total to the particular and precise detail.* The end-result of this process of differentiation is an organized body of "clear and distinct ideas"—that state of affairs so dear to the mind of the skilled logician. But it is far from the condition with which the growth process starts Sharpness of outline is what we end with, not what we have at the beginning. The act of perceiving is normally initiated by a dim general awareness of the object; it is only as this continues to act upon the observer that its internal "structure" emerges.

In the light of this conception many mathematical commonplaces acquire a new meaning. Euclidean geometry, e.g., is a masterpiece of reasoning, but it would be far truer to the facts of genetic psychology if the order in which it is commonly presented were completely reversed. Its logic is atomistic or elementaristic,[1] i.e., it begins with the most highly refined and mature abstractions, such as "definitions" (note the term with its suggestion of optical focusing!) of *point* and line, rather than starting with more massive and "natural" percepts like cubes, surfaces, etc. Psychological experimentation indicates that a "point" is a fairly late and high-grade achievement of one's spatial understanding. Our visual-tactile world is not originally made up of *points*—instead, these emerge from it. Paradoxical as it may seem in the light of the usual placement of courses, solid or tri-dimensional

[1] The following clipping from the humorous column of a teachers' journal indicates that many persons have become aware of the artificial and "absurd atomism" implied in many of the textbook problems of an older day.

"Arithmetic is a science of truth," said the professor earnestly. "Figures can't lie. For instance, if one man can build a house in 12 days, 12 men can build it in one."

"Yes," interrupted a quick brained student, "Then 288 will built it in one hour, 17,280 in one minute, and 1,036,800 in one second. And I don't believe they could lay one brick in that time."

geometry is the source of all later spatial analysis which ends in, but does not proceed from, the strange entity that "has" location without extension. It has even been argued that division is a more primitive arithmetical operation than addition.

3. *The properties of parts are functions of the whole or total system in which they are imbedded.* In perception this principle is clearly observed by the fact that a grey square upon a blue ground looks yellowish and the *same* grey patch on a yellow field appears bluish (color induction or "contrast"). In Fig. 2(c), most persons "see" a square and a hexagon in contact (presumably because the structural organization of the drawing favors this response), but the capital letter *K*, which is just as much present in substance, is usually not discerned. To isolate a familiar portion of the alphabet in this situation requires the segregation of two markedly dependent parts of two strongly unified "figures." Frequency and repetition cannot account for this phenomenon, for even the most experienced geometer has encountered a *K* more often than he has seen these elementary patterns in contact. The external arrangement as given compels a corresponding internal organization.

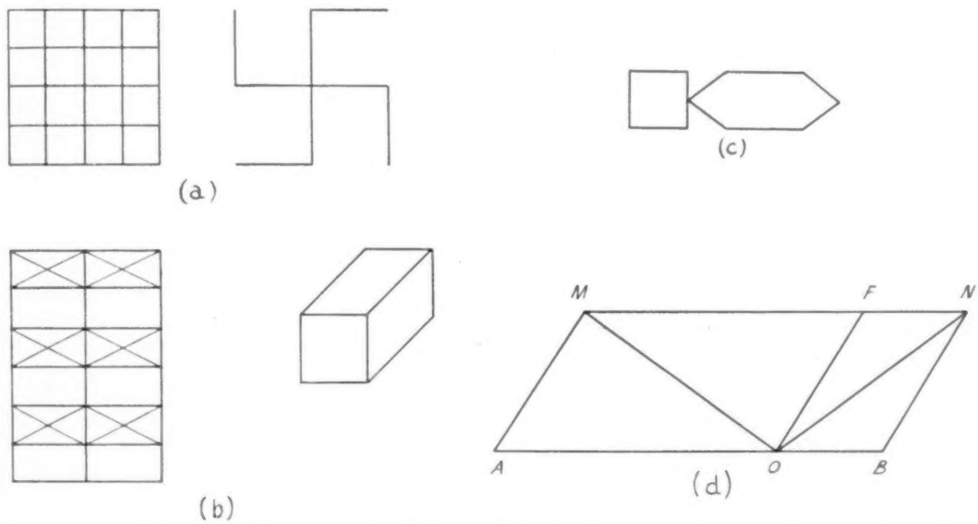


FIG. 2. How "field forces" govern what is discriminated. Can you isolate the swastika and the "box" in their companion figures on the left?

Fig. 2(d) is even more impressive because it exhibits some of the mechanism underlying an illusion. Most persons, if asked to compare the diagonals *MO* and *ON*, will unhesitatingly declare *MO* the longer. Actually the two are drawn of equal length. The effect is apparently traceable to the larger rectangle *MAOF* which causes the observer to "see" (not to "infer" in the usual logical sense) its diagonal as greater than that of the noticeably smaller rectangle *FOBN*. The "illusion" is partly overcome by erecting a perpendicular at *O*, thus minimizing the unanalyzed and unequal influence of the two major areas. A better example of the Gestaltist's claim that a line is functionally a derivative of a plane could hardly be found.

It seems probable that the meaning of a number in series is a special case of membership-character being conditioned by its role in some structure. Thus, the number "364" is comparatively meaningless in isolation. Conceptually, however, its fuller meaning is necessarily derived from some schema, such as "less than

400," "between 350 and 400," "nearer 400 than 300," etc. In the case of lightning calculators, most numbers have acquired some such "individuality" as this—a fact which contributes something to an understanding of their ability.

With these three generalizations as one's conceptual tools, it is surprising how many obscure phenomena swiftly become more intelligible. A number of years ago, while examining the arithmetical errors of college students, I noticed that a mistake was more likely to occur when a larger number was being added to a smaller one than in the converse case. Thus, $9+7$ and $7+9$ both make 16, but addition errors are decidedly more frequent with the latter combination. The rule has also been statistically verified for two-place numbers and I am inclined to believe it holds for fractions and any other number combinations. If we view the addition of two quantities as a simple case of completing an indicated total, then this observation is brought under the head of "closure" phenomena. In any language completion test, gaps are to be filled in, and the test's difficulty is roughly proportional to the number and extent of the gaps involved. This "totalizing effect" is seen in the figures below. A "circle" with three-fourths of circumference visible is easily seen as a full circle at a slight distance from the eye—a fact occasionally used by the oculist in visual testing. An arc equivalent to a quarter-circle does not lend itself so readily to the "restoration" of the entire circle (cf. Fig. 3). *Pari passu*, when one

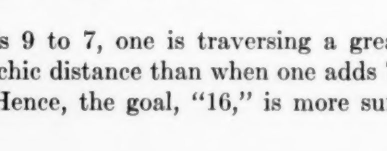
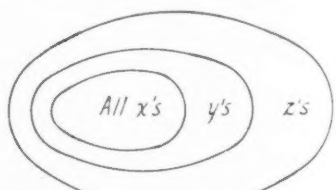


Fig. 3. Closure effects are easier, less ambiguous, and more impressive the less the perceptual "filling" required.

adds 9 to 7, one is traversing a greater psychic distance than when one adds 7 to 9. Hence, the goal, "16," is more surely reached in the latter instance. Since there is no inherent difference in the difficulty of the two symbolic number combinations, the variation in accuracy must be traced to the forces behind the schematic structure by which they are represented in the organism.

If this explanation makes concepts obey the same laws as percepts, that has its source in the conviction that thinking and reasoning are dependent upon the processes of perceiving. The latter activity is nearer to the concrete situation than the former, and ordinarily involves less of the inferential closure represented by the dotted portions of Fig. 3. But in the case of "Euler's circles" as used in elementary demonstrations of formal logic, one literally "sees" how intimately syllogistic proof is linked to direct sensory perception of the basic pattern. It seems that the famous Swiss mathematician of the eighteenth century was once a tutor by correspondence to a dull-witted Russian princess and devised this method of convincing her of the reality and necessity of certain relations established deductively. Thus, the syllogism, can be

All x is y;
All y is z;
$\therefore$ All x is z



tested for its truth or falsity if diagrammed as indicated. Here the concentric "ellipses" reproduce the essentials of the situation so faithfully that the answer becomes a matter of "mere inspection."

This process of making an organism aware of the conditions governing the phenomena to which it is reacting is essentially what is meant by the "insight" experience. Rightly construed, insight is not a peculiarity of the "higher" rational functions, but a process necessarily occurring at all mental levels. Simple ad-

justments of bodily position, such as lifting the foot to go up a step or bending the knees when accepting an invitation to sit down, constantly exhibit a low but decidedly real level of insight. In every case there is a re-arrangement of an action-pattern as a result of changing forces in the "field." If the new organization fits the needs of the situation, the neural stresses and strains return to a state of relative dynamic equilibrium and the "problem" is said to be "solved."

A simple geometrical example will suffice to clarify this all-important point. Given Fig. 4(a), a high-school sophomore is asked to find the area of a circumscribed square, knowing only the radius of the inscribed circle. A question or assignment such as this normally produces a mild tension which does not vanish until an answer satisfactory to the organism is achieved. If one examines carefully what occurs, it is plain that a state of mild bewilderment sets in which lasts until a configuration emerges. Fig. 4(b) is typical of a wide range of educational situations. The pupil is blocked as long as he sees the key item (the radius r) *fixed* in its first position; but as soon as he *shifts* it mentally to position a or b, its properties are *transformed*—it is no longer a half-diameter, but is now half of a side of the square. This reorganization accomplished, the response $A=(2r)^2$ is immediately and confidently made. The delay involved is almost entirely taken up by the time required to bring Gestalt principle No. 3 (above) into action.[2]

Another basic conception of Gestalt theory which promises to be useful in mathematical instruction is the idea of transposition. This had its origin in the common observation that a musical melody is not the same as the sum of the separate tones that presumably comprise it. "The Star Spangled Banner" can be sung by bass, soprano, and other voices, or played by various orchestral instruments in different "keys." In none of these instances need any of the individual tones be alike, and yet the common pattern or melodic sequence is easily recognized. The melody is the whole which is transposed and *transposable* from one situation to another.

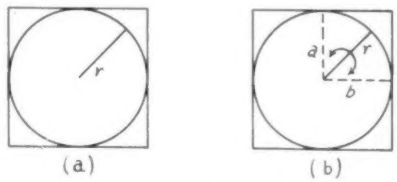


FIG. 4. How insight operates with quantitative relations.

Is it not conceivable that most of the "abstractions" with which mathematics deals are of this nature? The ratio between the diameter of any circle and its circumference is expressed by the constant π, and this relation is transposable whether the actual concrete materials that constitute the circle are made of copper, rope or carbon and liquid particles. In the case of the equation—which is plainly the heart of most mathematical operations—we see how the procedure is built around the possibility of preserving "permanence amid change." One may think of the equation as a step-wise expanding or contracting pattern that preserves invariant the condition of equality or identity throughout the various transformations that may be legitimately applied to it.

This situation was long ago appreciated by gifted thinkers and accounted for much of the esteem in which mathematical method was held. However, it would be a great mistake to assume that other aspects of nature and behavior are lacking in this respect.[3] Even a goldfish (who is known to be less intelligent than the humble

[2] Alternative and equally simple solutions are also possible.—On one occasion a mathematics teacher, who was listening to an *oral* exposition of this example, insisted the answer was wrong because he had heard and interpreted $(2r)^2$ as $2(r)^2$. It is significant that erroneous as well as correct solutions are equally intelligible under this principle.

[3] Consequently geometry cannot be the only area in which rigorous proof is possible. No proof can rise above the assumptions and hidden theorems upon which it rests.

cockroach) reacts appropriately to a group of "equivalent stimuli." Thus, suppose he has learned that food is to be found behind that light which is intermediate in intensity among three sources of illumination in his field. Now change the absolute brightness of the three bulbs by doubling or halving the "intensity" of the current. In either event, the goldfish swims unhesitatingly toward that light which is relatively in the "middle" of the triad. This response—found widely in all species—is difficult or impossible to explain on the basis that learning is specific, i.e., restricted in its effectiveness to the precise stimulus-object employed. This interpretation itself must be false as it is killed by its own hyper-specificity. No two learning situations can be alike in *all* respects; the likeness is found in the organization of the wholes and not in the substance of the pieces. A wooden table is functionally more like a metal table than it is like a wooden chair. The way things are put together determines the attributes of the system thus established.

It must be obvious from these illustrations that mathematical and psychological research have more in common than is usually realized. The field and organismic approaches to behavior have more than a purely physical and biological connotation, and point set theory is heavily used in systems of "topological" psychology. Much of this is the inevitable and desirable consequence of the unity of scientific thought. The contributions of mathematical technique to psychological advance have been most impressive, but the reverse type of obligation has rarely been incurred, presumably because there was so little to borrow! Perhaps if mathematics teachers act upon the recognition that the content of their discipline has to be re-discovered and created *de novo* by every learner, they will have provided themselves with the one tool needed to make the Grand Tradition of rigorous thinking influential in the lives of our people.

Our Infinitesimal Nature

I know of nothing which acts as such a powerful antidote to that which I venture to call "opinionatedness," as a study of mathematics. To know that the light from solar systems far larger than our own has been thousands of years in reaching us, gives us an idea of our infinitesimal nature, in comparison with space about us, that can come only with a study of the science that it is ours to teach. A bacillus in our veins, so small as to be invisible through a powerful microscope, is a giant compared with ourselves in our relation to this space in which we live. Our doubts, our beliefs, our hopes, our fears are all so trivial, so infinitesimal, so like a lost electron in our solar system, as compared with our relative importance in the universe as revealed to us by the calculations which mathematics brings to bear upon the great problem! Cowper wrote well when he put in verse the words,

> "God never meant that man should scale the heavens
> By strides of human wisdom,"

and even the mathematics of youth confirms the thought—David Eugene Smith.

Calculating Machines and the Mathematics Teacher

By Evelyn M. Horton

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The field of the calculating machine is one which should be of interest to every mathematics teacher who is concerned with equipping the student with every possible device to cope with the present trend in complicated educational and commercial figuring.

Figures touch the lives of the people more closely today than in any previous age. The history of a country in ancient times was, generally speaking, an account of the activities of the ruler in conquest. He went to war and returned with lands and people added to his domain. Today history is written almost completely in terms of figures—"So many millions of dollars were voted by Congress for flood relief." "In England it is reported that so many millions of £'s worth of chinaware was manufactured in the famous potteries at Hanley in the Midlands." "The French have a war debt of so many millions of francs." To even understand the day's news in the papers one must know how to interpret figures correctly because tables, graphs and statistical data are interspersed freely in the reading matter.

So, with this in mind, what can be more necessary to aid the present growing generation to be useful and intelligent citizens than to supply them with a knowledge, at least, of what calculating machines can accomplish, and, better still, an actual experience in learning the simple processes of mathematics by the machine method?

There is hardly a position in any capacity where the ability to use one or more of the various types of calculating machines would not be advantageous; in some cases, it is absolutely necessary.

In order to clarify the term "Calculating Machines," it is necessary to give an explanation. There are four types of mathematical machines in regular use today.

Type 1. The Listing Machine
Type 2. The Bookkeeping Machine
Type 3. The Key-Punch and Tabulating Machines
Type 4. The Calculating Machine

1. *The Listing Machine*

The Listing Machine adds, and in some cases subtracts, and prints the results on paper. It is made in two general mechanical types. The first, known as the Full Keyboard Type and the second, as the Ten-Key Type.

The Full Keyboard Type has rows or "banks" of keys as they are called, running up the keyboard from 1 to 9 and as many across as happens to be the capacity of the machine.

The Ten-Key Type has but ten keys in all, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0.

The illustration below shows the two keyboard types:

9	9	9	9	9	9	9	9
8	8	8	8	8	8	8	8
7	7	7	7	7	7	7	7
6	6	6	6	6	6	6	6
5	5	5	5	5	5	5	5
4	4	4	4	4	4	4	4
3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1

Full Keyboard Type

7	8	9
4	5	6
1	2	3
	0	

Ten-Key Type

To add the number "125" on the first type of keyboard, the operator would depress the 1 key in the third column from the right, or the hundreds column, then the 2 key in the second column from the right, the tens column, and then the 5 in the first or units column. These keys stay depressed on the keyboard and give an opportunity to check the amount entered before throwing it into the answer wheels. If it is correct the handle is pulled if it is a hand operated machine, or the

motor bar pressed if it is electrically driven. Three things happen at this operation. The depressed keys return to their original position, the number is entered in the answer wheels and the first entry is printed on the adding machine tape.

The operator proceeds in the same manner with the second and succeeding numbers until the last has been reached. The motor bar or handle is then operated one extra time, as you would draw a line under a column of figures before entering the total, and the key marked TOTAL at the side of the keyboard (the position of this key varies with different makes of machines) is pressed. The bar is pressed again

THE BURROUGHS LISTING MACHINE
An Illustration of the Full Keyboard Type of Machine

then and the sum with an asterisk or some other symbol to indicate that it is the total, is printed on the tape.

If a number of pages are to be added and the total carried forward, there is a subtotal key which gives the total of page 1 then adds the figures of page 2 to that and so on until at the end the TOTAL key gives the grand total of all the pages.

In the case of the second type of machine a "touch method" of operation may be developed with a little practice and patience.

If "125" were to be added on this machine the 1 key would be depressed but it would not remain down but return to its normal position. An arrow indicator above the keyboard would move from a 0 position to 1, showing that one digit had been placed, then the 2 would be depressed and the arrow would move to the second or tens position and then the 5 and the arrow would move to the third or hundreds position. This indicates that a number of three digits have been set up.

There is no way to check the entries on this machine, only after the addition bar has been pressed and the number entered in the adding wheels and printed on the tape can an error be located.

All 0's must be pressed here, too, whereas on the other type of listing machine, the column that has the 0 is skipped and the 0 is printed on the tape automatically.

This Ten-Key Type of listing machine requires more practice and skill to operate, but greater speed may be developed because of the possibilities of the touch feature.

The Listing Machine does not require any specialized training and its use may be taught in a few minutes. Facility comes with practice and it is a machine that could be taught easily as a part of regular school work.

2. *The Bookkeeping Machine*

This type of machine, when used in business houses, does "extension" work. By this is meant such calculations as "Find the cost of 225 gross of pearl buttons at $3.00 a dozen less discounts of 10%–5%."

In brokerage houses of Wall Street it is used to figure stock transactions in such cases as, "What is the cost of 25 shares of Consolidated Film Preferred at $17\frac{1}{2}$ per share plus a commission fee of $6.00?"

The machine is operated as a typewriter with the results being figured automatically by the machine and typed on the bill or ledger sheet as the case may be.

In this class of machine falls the

Burroughs Bookkeeping Machine
Burroughs Moon-Hopkins Machine
The Underwood Elliott Fisher
The Monroe Bookkeeping Machine

The Bookkeeping Machine is one that requires specialized training which may be procured at the schools run and oper-

ated by the concerns selling the equipment. This is not a machine to be thought of in terms of teaching in the public schools. It has a very specialized field and provides a trained position in the class with a stenographer or telegrapher.

Burroughs Moon-Hopkins Bookkeeping Machine

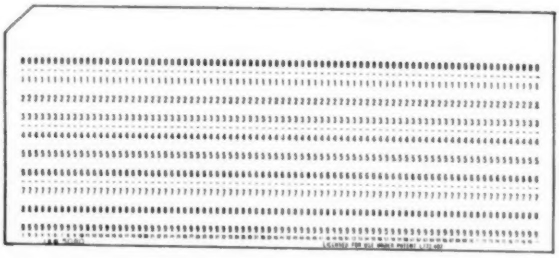
3. *The Key-Punch and Tabulating Machine*

This equipment is composed of two machines. The operator may learn both or only one. They are the Key-Punch Machine and the Sorting and Counting Machine. To briefly outline the field,—there is a card $7\frac{3}{8}'' \times 3\frac{1}{4}''$ (marked off into 80 columns) and numbered vertically. The material to be tabulated is coded on the original blanks in such a manner that it may be entered upon the card by punching out the number that corresponds with the number of the code on the blank. For example, an insurance company might have the first column of the card to represent the states, each state being coded by a number, Alabama as 1, etc. The second column might represent the district and that might be 8, the next might indicate male or female, 1 and 2 respectively, the next might be age at the writing of the policy. This would need two columns and might be 49.

After the numbers of the code have been entered on the original blank it is handed to a key-punch operator who slips a blank card into the machine and by striking the key corresponding to the number of the code takes out the numbers, leaving a series of small round holes.

Then when a study is to be made, for instance for some educational study, it might be interesting to know how many woman teachers in the various states over the age of 25 carried different types of insurance, with the Metropolitan Insurance Company.

To arrive at these figures the sorting and counting unit of the machine would be so set as to have the electricity go through those little holes that had been punched from the code. Then all the cards would be placed in the machine and the electricity turned on. Automatically it would sort the cards out into the various compartments at the front of the machine. Each compartment is equipped with a counter and at the end of the run the totals would be all ready to be taken off.



An Illustration of the Key-Punch Card

Where a vast amount of data is to be tabulated and studied in various ways, this machine is most admirably suited to the work.

The International Business Machine Cor-

poration, Tabulating Machine Equipment Division puts out this equipment. Operators are generally trained in the business houses where they work. Girls are most often used on the punching machines and boys trained on the sorter and counter, as this requires more mechanical skill than is generally attributed to girls.

This, also, is not a type of equipment that could be taught in school, but it represents a field for young people to enter.

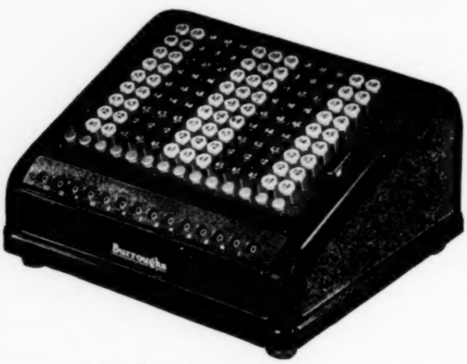
4. *The Calculating Machine*

This is the type of machine in which there is the most interest in connection with school work. Calculating machines fall into two classes:

The Key-Driven Machine
The Non-Key Driven Machine

The Key-Driven Machine

A. The Burroughs Calculator
B. The Felt and Tarrant Comptometer.



THE BURROUGHS CALCULATOR
An Illustration of the Key-Driven Machine

One can add, subtract, multiply, divide and figure square root on these two machines and they are identical in operation. If a person can operate one, he can operate the other.

These machines require a trained operator. It takes about 200 hours of class work in school to learn to run the machine with sufficient skill to be able to hold a position.

The reason for this is made known when the term "Key-Driven" is explained. By this is meant that the instant the operator depresses a key on a Calculator or Comptometer the result is registered in the answer wheels and there is nothing that can be done about it if it is not correct except to clear the machine and begin again.

Where this much accuracy is required it can be seen easily that a skilled technique must be developed to arrive at the correct result the first time. This takes hours of practice.

All the processes on the key-driven type of machine revert to some form of addition. Addition, of course, is just the process of placing one number after the other in the machine.

Multiplication is continued addition. If 25 is to be multiplied by 2, the operator places his fingers on the 2 in the tens column and the 5 in the units column, and simultaneously depresses them twice. If the numbers to be multiplied were 25×62, he would place his fingers on the 2 and the 5 as before and depress them twice. Then slipping each finger over one place to the left so that the 2 would now be in the hundreds column and the 5 in the tens, leaving the units column vacant, he would depress these keys 6 times and the combined result would be registered in the answer wheels.

The mechanical reason for this is easily seen if you will place the figures on paper and work out the multiplication; e.g.,

$$\begin{array}{rll} & 25 & \\ \times & 62 & \\ \hline & 50 & =25\times 2 \\ & 150 & =25\times 6 \text{ (One place to the left because we are really multiplying 60, not 6)} \\ \hline & 1550 & \end{array}$$

Subtraction is a matter of "negative" addition. Each key on the machine has two figures on it, a large one and a small one. The small ones are the negative or complementary figures. The sum of each key is equal to 9. If the operator wishes to subtract 25 from 100, he would add the

100 in the machine in the usual way and then depress the small 2 in the tens column and the small 4 in the units column, the rule being the subtrahend is always reduced by one. The result in the answer wheels will be 75 and if the large numbers on the keys having the little 2 and little 4 are examined they will be found to be 75. What happens is that the complement is added in. In other words, by using the small 24 he adds enough to make up the difference between that and the 100 he started off with. This sounds very complicated but is really easy in actual operation, because all the operator has to remember is to take the number to be subtracted out with the small figures, one less.

Division is a matter of continued subtraction, but too involved to go into here.

The process of addition takes the longest time to master for speed and accuracy, as it is executed entirely by the "touch" method. The operator is trained to know that the odd numbered keys, 1, 3, and 5, are hollowed out in the center and the even ones, 2 and 4, are flat. This permits the development of a sense of touch to guide the student operators as they are not allowed at any time to look at the keys in addition.

As the stretch from the 1 key to the 9 key (a distance of about $6\frac{1}{2}$ inches) is too great a span to develop a touch method on, only the keys from 1 to 5 (a distance of $3\frac{1}{2}$ inches) are used. To procure the numbers over 5, combinations of the lower figures are made. The 6 is arrived at by pressing the 3 key twice, the 7 by using the 4 and 3, the 8 by using the 4 key twice, and the 9 by combining the 5 and 4. Ten of course, is just the 1 in the second or tens column. Only the first and second fingers of the right hand are used in addition, even if the figures to be added are in the millions.

The Monroe Calculator
Hand Operated Model
An Illustration of the Non-Key Driven Machine

One of the greatest advantages of this type of machine is the speed which can be developed by "touch addition," as it enables the operator to keep his eyes on the figures and saves time in looking backward and forward from the figures to the keyboard. The eye-strain that is relieved is also a factor of inestimable importance.

This is not a machine to become "just familiar" with. It is one that only fulfills

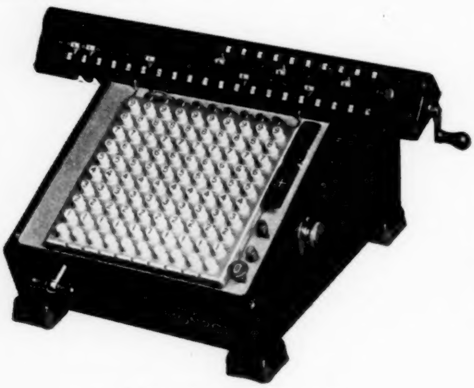
its purpose when operated by a skilled operator.

The key-driven machine has been placed in schools in the past and been taught successfully. There are also a number of private schools and schools run by the companies manufacturing the machines which train operators. These courses take from 4 to 6 months during a regular office day.

This machine, if taught correctly, has a decided field in the schools, and supplies the graduate with a semi-skilled position that can be turned always to use in the business world as a means of livelihood.

The Non-Key Driven Machine

This type of machine is made by many companies. A few of the better known machines are:



MONROE CALCULATOR
Electrically operated model

A. Monroe Calculator
B. Marchant Calculator
C. Frieden Calculator
D. Mathematon
E. Mercedes

These machines are made in various models; some are hand-operated, and some electrically driven. No matter which type you learn to operate, it is a matter of only a few minutes to transfer that knowledge to some other type. They do not require the period of instruction that the key-driven machine does, for the simple reason that the numbers are placed on the keyboard before they are "run" into the machine, and the operator has the opportunity to make any correction he wishes before the computation reaches the answer wheels.

The actual operation is very much like that outlined for the key-driven machines, except that subtraction is done directly. That is, complementary figures do not have to be used.

Calculating machines, both the key-driven and non key-driven types, might well be the standard equipment of any progressive school to be used not only in the office practice classes, but wherever mathematics is taught. The students thus trained can be also a great asset to the school office and the work done there serves as practical experience.

Companies of all varieties use the Key-Driven and Non-Key-Driven equipment; railways, insurance companies, shipping firms, wholesale and chain stores, oil companies, practically every kind of business organization that we see conducted.

In this article I have tried to bring to the attention of the teacher a brief explanation of the various types of calculating machines together with the necessity for, and the possibilities of machine computation as related to the school and the business world.

There is a vast field for the use of mechanical computation also in the graduate schools of the colleges where the candidates for the Master's and Ph.D. degrees find the ability to operate calculating machines necessary to their successful carrying on of any research they may attempt in a statistical line.

It is therefore a matter of concern to the teachers of mathematics to look into this question and examine the value of such instruction to the student, whether he enter the business world or continue on with his advanced education.

Illustrations supplied by the courtesy of: The Burroughs Adding Machine Co., The Monroe Calculating Machine Co., The International Business Machines Corporation, Tabulating Machine Division.

Linkages[1]

By Joseph Hilsenrath
Class of 1937
New Jersey State Teachers College at Montclair, N. J.

There is probably no construction in geometry which is so readily acceptable as that of drawing the straight line. Although Euclid postulated this construction in 300 B.C., it was not until over 2000 years had elapsed that a mechanical device was invented for making the construction. Euclidian geometry is based upon two fundamental postulates: first, that it is possible to draw a straight line between two points, second, that a circle can be drawn with any point as center and any line as radius. The second construction (postulate 3) is easy to make. Take any object, however irregular its shape, preferably flat but not necessarily so; choose any point on its surface, such as A, and thereafter cause that point to remain fixed at the required center in the plane. Choose another point B at the required distance from the first; cause the second point to move freely about the first, and the locus described by it will be a circle with A as center and AB as radius. (See figure 1.)

The drawing of a straight line (postulates 1 and 2) is not so easy. Neither Euclid's postulates nor his definitions point to a method. To overcome this difficulty, the text-book writers say that the first and second postulates of Euclid postulate the straightedge. But surely that is begging the question. If the line is to be straight, the straightedge used to construct it must have a straight line edge. And if it is not possible to draw a straight line by any other means, how was the straightedge made originally?[2]

[2] Kempe, Alfred Bray, "How to draw a straight line" page 2. (London, Macmillan and Co. 1877.)

The problem, then, is to devise a mechanism which will draw a straight line, but will not necessitate the use of a straight line in its own construction. Although mathematicians had been using the straight line for many centuries, it was the pressure of practical mechanics—particularly the invention of the steam engine—that was instrumental in stimulating research that resulted in the first geometrically rigorous description of the straight line. Because its practical applications are now outmoded, this branch of mathematics, unfortunately, has not received the attention which it merits.

One of the first to realize the importance of a mechanism which would trace a straight line was James Watt, the inventor of the steam engine. Dissatisfied with the limitations of the prevailing methods, especially with the straight-edged guides, he invented, in 1784, an apparatus which gave an approximate straight line motion. The mechanism, in its simplest form consists of two equal bars, HE and JF, (See photo 1) rotating about two fixed points H and J. The two free ends, E and F, are joined to a third bar, the midpoint of which is G. The loci of E and F are, of course, two circles with opposite curvature; the locus of the tracing point G is called Watt's Curve.[3] It is

[3] The equation of Watt's Curve is (in polar coordinates)

$$r^2 = b^2 - [a \cdot \sin\theta - (c^2 - a^2\cos^2\theta)^{\frac{1}{2}}]^2,$$

θ varying from 0 to π. $HE = JF = b$, $EG = FG = c$ and the distance from H or J to the nodal point equals a. Watt's curve becomes a lemniscate of Bernoulli when $c = a$ and $b = a\sqrt{2}$. For Watt's Curve see "Spezielle Algebraische und Transzendente Ebene Kurven" by Gino Loria, vol. 1. pages 278–279.

[1] The words: linkage, conicograph, and other technical terms used in this paper were coined by the British mathematician, James Joseph Sylvester in the Proceedings of the Royal Institution of Great Britain, Vol. 7, 1873–1875, pp. 179–198.

a FIGURE OF EIGHT curve reducible to a lemniscate. Certain portions of the curve near the nodal point (crossing point) are approximately straight. This simple mechanism was successfully employed in many engines of short stroke. A longer stroke caused the point G to describe too much of the lemniscate, thereby deviating appreciably from the required straight line. When the instrument deviated but little from its mean position the result was close enough for practical purposes. The relative size of the bar EF is immaterial, but the distance between the two fixed points should be such that when HE and JF are parallel, the bar EF will be perpendicular to them. This will insure an approximately straight line over a greater portion of the path.[4]

The Russian mathematician Tchebicheff of the University of St. Petersburg (Leningrad) applied himself vigorously to this problem, but was not successful in solving it. He did contribute, however, to the mathematical literature on the subject, and also perfected a mechanism which gave a closer approximation than that of Watt. In photo 2 the links are so proportioned that

$$JF/5 = HE/5 = EF/2;$$

the tracing point, G, is the mid-point of EF; the distance between the two fixed points J and H is twice that of EF. The locus of G is a flat closed curve, the bottom portion of which is an approximate straight line.

Another type of approximate straight-line motion is that of Richard Roberts (a machinist of Manchester, England), shown in photo 3. This "three-bar motion" consists of two equal links, CF and BE, fixed at C and B. The three ends are jointed to a triangular piece which carries the tracing-point G. $FG = EG = CF = BE$, and the distance between the fixed points, C and B, is twice that of FE. The portion of the path between the two fixed points, B and C, is an approximate straight line.

In 1864, A. Peaucellier, an officer of engineers of the French army, proposed, in the NOUVELLES ANNALES DE MATHEMATIQUES,[5] the problem of conversion of circular motion into rectilinear motion. He did not furnish his solution, and this discovery was forgotten until, in 1871, there appeared in a Russian journal,[6] the same solution by L. Lipkin, a student of Tchebicheff at the University of St. Petersburg. Tchebicheff was so delighted with the success of his pupil that he was instrumental in obtaining for him a substantial award from the Russian Government. The publicity created, helped recall the work done by Peaucellier in 1864, and in 1873 Peaucellier published his solution in the same journal in which he had proposed it, nine years previous.[7] Lipkin's independence and Peaucellier's priority were subsequently acknowledged, and Peaucellier received the "Prix Montyon," the great mechanical prize of the Institute de France.

Peaucellier's CELL, so called, consists of four equal links pivoted together to form a rhombus, $IACB$ (fig. 2); to two opposite vertices, A, B, are pivoted two equal links, AD, BD, joined together at D. It is obvious that the three points I, C, D, are collinear in all positions of the instrument. Its most remarkable property is that the product of the distances CD and ID is a constant.

Draw the line ICD and the other diagonal of the rhombus which meets it in

[4] For a short interval on either side of the origin, there is little change in slope; the portion of the curve within this interval is utilized as the approximate straight line.

[5] Peaucellier, A. "Lettre au rédacteur des Nouvelles Annales de Mathématiques." Nouv. Ann. de Math., series 2, Vol. 3, 1864; pages 414–415.

[6] Lipkin, L. "Über eine genaue Gelenk-Geradführung." *Bulletin de l'Academie Imperiale des Sciences de Saint-Pétersbourg, Bulletin*, 16, 1871; columns 57–60.

[7] Peaucellier, A. "Note sur une question de géométrie de compas." Nouv. Ann. de Math. series 2, Vol. 12, 1873; pages 71–78. This article was translated by Prof. W. D. Marks in the Franklin Institute, *Journal*. vol. 77, 1878; page 361.

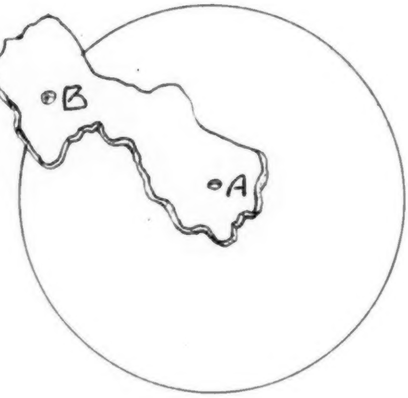


Fig. 1

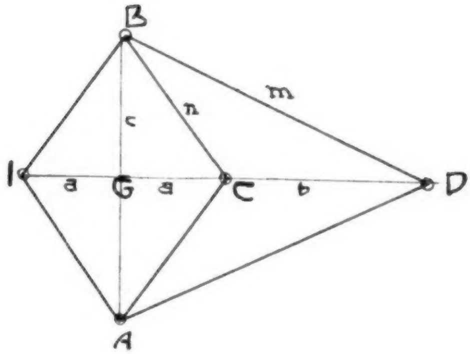


Fig. 2

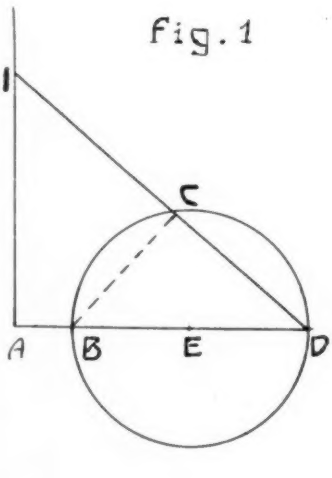


Fig. 3

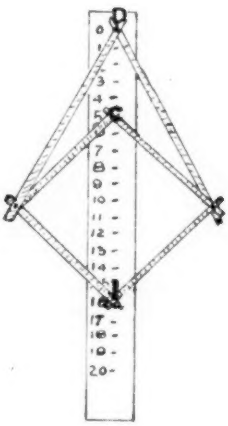


Fig. 5

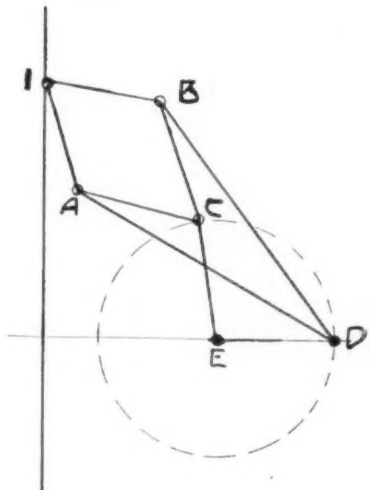


Fig. 4

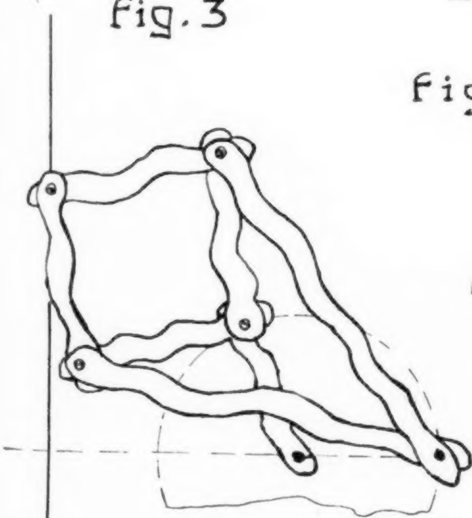


Fig. 6

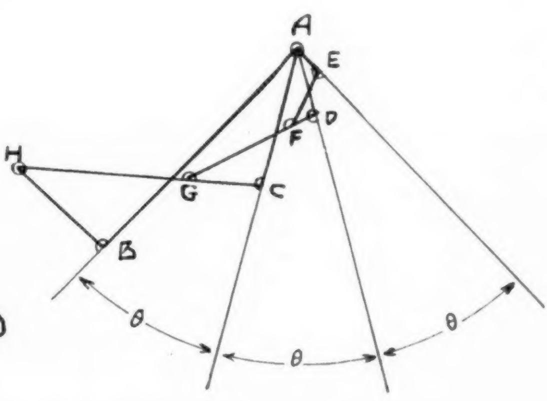


Fig. 7


1
WATT'S
APPROXIMATE STRAIGHT-LINE MOTION
2
TCHEBICHEV'S
APPROXIMATE STRAIGHT-LINE MOTION
3
RICHARD ROBERTS'
APPROXIMATE STRAIGHT-LINE MOTION
Peaucellier's Cell
EXACT STRAIGHT-LINE MOTION
5
Peaucellier's Conicograph
6
BRICARD'S
EXACT STRAIGHT-LINE MOTION
Richard Roberts' Conicograph
Kempe's Tram-Motion
9
Kempe Angle Trisector

the point G. In the rhombus, $IG=GC$, and the angle BGC is a right angle.

Let $BD=m;\ BC=n;\ IG=GC=a;$
$CD=b;\ BG=c.$

then $m^2=c^2+(a+b)^2$

and $n^2=c^2+a^2$

therefore, $m^2-n^2=b^2+2ab=b(b+2a)$

or $m^2-n^2=CD\cdot ID.$ (1)

Since m and n are both constants, $CD\cdot ID$ is constant for all positions of the CELL. This property enables its use in the drawing of the straight line. Figure 5 shows a little device that has been found helpful in convincing the more skeptical members of a high school audience of the truth of this remarkable property of the Peaucellier CELL. The point D of a CELL is fixed to the zero point of a graduated scale, the points C and I will assume different positions along the scale as the CELL is deformed. In this instance $m=12$, $n=8$; if what we proved in equation (1) is true, the product of the distances of C and I from D should always be equal to 80. By moving the CELL so that $CD=4$, we see that $ID=20$; the product is 80. When CD is 5, ID is 16. When CD is 8, ID is 10 etc. This should be convincing enough for the general high school audience.

In figure 3, we have a circle with its center at E and diameter DB, which is extended to a point A; at A a perpendicular is erected. Choose any point, I, on this perpendicular and connect it with the farther extremity of the diameter. The line ID, thus formed, intersects the circle in C; join C with B, making a right angle BCD. Since IAD is a right triangle by construction, we have two right triangles, IAD, BCD, with a common acute angle. The triangles are similar and $ID/BD=AD/CD$ or $ID\cdot CD=AD\cdot BD$. But AD and BD are constants; therefore the product $ID\cdot CD$ is a constant. Since I was any point on the line, the relationship holds for all points on the line.

The problem, then, is to make two points move so that the product of their distance from a third fixed point is a constant. (This is exactly what our CELL does.) If one of these points is made to assume successive positions on the circumference of a circle, which contains the fixed point, then the other point will describe a straight line perpendicular to the diameter which contains the fixed point. Fix the point D of a CELL on the circumference of a circle (fig. 4). Constrain C to move in this circumference by the addition of a radius-bar CE. The point I will trace the straight line. Adopting the terminology of inverse geometry, THE INVERSE OF A CIRCLE PASSING THROUGH THE CENTER OF INVERSION IS A STRAIGHT LINE PERPENDICULAR TO THE DIAMETER OF THE CIRCLE WHICH PASSES THROUGH THE CENTER OF INVERSION.[8]

If the point D be fixed either inside or out of the circle, the inverse of the circle will be another circle. As D is moved closer to the circumference, the arc described by I approaches more nearly that of the straight line. When D is infinitesimally distant from the circumference, the point I describes an arc of a circle with infinite radius, which is a straight line. By varying this distance it is possible to draw circles with large radii. The same thing is accomplished in photo 4 by varying the length of the radius-bar CE.

In photo 5, instead of fixing the point D, we fix C and cause D to move around the circumference of the circle. The point I, now, traces out a hypo-cissoid, cissoid, or hyper-cissoid according as the distance between C and E is: greater than, equal to, or less than the radius ED. When $CD=DE$, the curve traced out is a Cissoid of Diocles.[9]

[8] Although the proof of this theorem by inverse geometry possesses many niceties, it was thought that the geometric proof given above would have more appeal to teachers of secondary mathematics. This proof has been used successfully with many High School mathematics clubs and classes.

[9] Roos, J. D. C. de "Linkages: the different forms and uses of Articulated links." pp. 56–57; Van Nostrand, N. Y. 1879.

In photo 6 we have another straight-line motion arrived at by Raoul Bricard[10] from one of a more general nature proposed by Harry Hart.[11]

$$FB = GC = a;\ BA = CA = b;$$
$$DB = EC = a^2/b;\ FG = c;\ DE = ac/b.$$

The point A describes the straight line which is the perpendicular bisector of the line segment joining the two fixed points F and G. This instrument is based upon the simple property of a straight line: *viz*, that it is the locus of a point which moves so that the difference between the squares of its distance from two fixed points is a constant.

In photo 7 we have a linkage for the tracing of a parabola. The portion of it that is due to Roberts[12] is the assemblage $IGFVE$. The point I traces out the "pear shaped" curve whose polar equation is the reciprocal of that of the parabola, when the point D is taken as the origin of the polar coordinate system.[13]

In the linkage in photo 8,

$$EA = ED = a;\ AC = CG = CD = b;$$
$$GF = FD = b^2/a.$$

The point G is fixed and the extra link IE is added (it completes the parallelogram $IGFE$) to cause ED to move always parallel to its original position. When this happens, the motion of the disc, diameter AB, is a cyclic motion inside of a circle of twice the diameter. The locus of the point C is simply a circle with center G and radius b. If a tracing point is situated at a distance x from C, the locus is an ellipse, the axes of which are $2(x+b)$ and $2(x-b)$. When $x = b$, that is, when the tracing point lies on the circumference of the rolling circle, the ellipse degenerates into a straight line of length $4b$. When the tracing point lies at the other extremity of the diameter, the line traced is perpendicular to the first and also of length $4b$.[14] The photo shows the points A and B which describe two mutually perpendicular lines. Any point between them, except C, will describe an ellipse. This model has proved itself especially apt for the illustration of locus problems, and for showing the relationship between the ellipse, circle, and the straight line.

One of the most interesting topics with high school students is the problem of the trisection of the angle. Much has been written on the efforts to do this with the compass and the unmarked straightedge and on the various solutions of this problem by other means. Photo 9 shows a linkage for the trisection of an angle invented by Alfred Bray Kempe. It is interesting to note, here, that Kempe was a London barrister whose specialty was ecclesiastical law. He became interested in linkages while attending a lecture on the subject given by Professor J. J. Sylvester on the twenty-third of January 1874. This was one of the now famous Friday evening lectures before the Royal Institution of London. Sylvester, himself, became interested in the subject in 1872 when Professor Tchebicheff, who was then his guest in London, called his attention to the work done by his pupil, Lipkin, to which we have already alluded.

In figure 7,

$$AB = HC = b;\ HB = AC = GD = a;$$
$$GC = AD = FE = a^2/b\ \ FD = AE = a^3/b^2.$$

Because their corresponding sides are proportional and they have the angle C in common, the two "contra-parallelograms" $ABHC$ and $ACGD$ are similar. Hence, the corresponding angles BAC and CAD are equal for all positions of the instru-

In photo 5 the curves are, reading from the bottom up, a hypo-cissoid, Cissoid of Dioceles, and two hyper-cissoids.

[10] Bricard, Raoul "Sur une nouvelle mode de description de la ligne droite au moyen de tiges articulées." *Compte Rendus de l'Acadamie des Sciences*, Vol. 120, 1895; page 69.

[11] Hart, Harry "On two models of parallel motion." *Proceedings of the Cambridge Philosophical Society*, Vol. 3, 1880; page 187.

[12] Since the photo of this linkage was taken, it has been found that there is no recorded authority for attributing it to Richard Roberts. de Roos refers to its originator simply as Roberts.

[13] de Roos. pp. 68–69.

[14] Kempe, A. B. "On Some New Linkages." *Cambridge Messenger of Mathematics*. Vol. 4, 1875, pages 121–124.

ment. The two figures $ADFE$ and $ACGD$ have the angle D in common, and their sides are also proportional; therefore, their corresponding angles, CAD and DAE are equal. As this instrument is set in motion there is generated about the point A three equal angles, BAC, CAD, DAE. If we extend the lines AB, AC, AE, until they are of equal length, the linkage provides a simple angle trisection. One could add any number of similar figures and thereby divide, or multiply, an angle as many times as is desired. In general a linkage of $2(n+1)$ links can divide an angle into n parts.[15]

At this point the objection is often raised that the construction of these instruments necessitates the assumption of a straight line; this, however, is not true. The linkages in the photos are constructed of straight pieces because the stock available was such; it was not necessary that they be straight. The only assumption that is made is: that if two points are fixed on a rigid body, they will remain at a constant distance from each other throughout. Furthermore, in the case of the Peaucellier CELL the only relationship between m and n is that they be different from each other; hence it is not even necessary to use a graduated ruler in its construction. In figure 6 we have a CELL constructed of irregularly shaped links. The straight line drawn by it is as accurate as that drawn by the one in photo 4.

Lest it be thought that mechanical delineation is the only field in which the subject of linkages has any application, I take the liberty to mention, however, briefly, a few of the many applications to which linkages can be put: the mechanical solution of equations, mechanical involution and evolution, mechanical transformation as applied to projective geometry, illustrations of the cubic transformation of elliptic integrals, algebraic transformation of a complex variable, mechanical calculators, etc.

[15] *Ibid.*, page 124.

Perhaps the best way to realize what interest this subject might have for high school students of mathematics, is to read a portion of what Sylvester has to say.

"He [Sylvester refers to himself, this being part of a footnote to his lecture at the Royal Institution of London] showed Mr. Garcia [a friend of his] the drawing of the cell left by Tchebicheff, and the next day was gratified by receiving from him a model constructed with a few pieces of wood, fastened together with nails as pivots, which, rough as it was worked perfectly, and drew forth the most lively expressions of admiration from some of the most distinguished members of the Philosophical Club of the Royal Society (not mathematicians, but naturalists, geologists, chemists, and physicists), when it was brought in with the dessert, to be seen by them after dinner, as is the laudable custom among the members of that eminent body in making known to each other the latest scientific novelties. Presently, after the speaker [Sylvester] exhibited the same model in the hall of the Atheneum Club to his brilliant friend Sir William Thomson of Glasgow, who nursed it as if it had been his own child, and when a motion was made to relieve him of it, replied, 'No! I have not had nearly enough of it . . . it is the most beautiful thing I have ever seen in my life.' This rude but rather invaluable model ought to be preserved in some physical laboratory as a historical relic. It served as an instrument by which the speaker [Sylvester] in every case gained immediate converts to the belief of the importance of Peaucellier's great discovery, whereas a mere geometrical diagram would have been as little regarded as a figure of the celebrated asses' bridge in Euclid at last so great is the difference of impression produced on the practical English mind by the *esse* and the *posse*—being told how a thing ought to act, and seeing it actually going. Considering the extraordinary conversions worked with Mr. Garcia's model, it would

not be unsuitable to write in letters of gold on the board attached to it which gives support to the two frail centers, the famous motto of Constantine—'In hoc signo vinces'."[16]

Such was the effect produced upon the members of one of England's learned societies. Although some of the above must be attributed to Sylvester's characteristic eloquence and enthusiasm, the subject was deemed important enough to attract contributions from such mathematicians of the time as Samuel Roberts, Arthur Cayley, Harry Hart, H. Brocard, Amedee Mannheim, W. W. Johnson, and more recently, F. V. Morley, and Arnold Emch. The writer's experience with numerous high school groups, both teachers and pupils, verifies the enthusiasm with which this subject is greeted on all sides.

[16] Proceedings of the Royal Institution of Great Britain, Vol. 7, 1873–1875, page 183.

NOTE. In the course "History and Appreciation of Mathematics" given to students majoring in mathematics at the State Teachers College at Montclair, special reports are required of each student on one of a list of topics suitable for presentation before a high school Mathematics Club.

This paper represents a more extended study on one of these topics. Mr. Hilsenrath became interested in "Linkages" through hearing the subject discussed at one of the meetings of the college Mathematics Club. He has prepared a set of models and has made talks on Linkages before Mathematics Clubs in many New Jersey High Schools—VIRGIL S. MALLORY, *Professor of Mathematics.*

Modern Preparation for College*

By Harold G. Campbell
Superintendent of Schools
New York City

Lamenting the uncertainty that jurists sometimes have of the ultimate wisdom of their decisions, Judge Cardozo once remarked upon the peace of mind that must come to the designer of a mighty bridge:

> The finished product of his work is there before his eyes with all the beauty and simplicity and inevitableness of truth. He is not harrowed by misgivings whether the towers and piers and cables will stand the stress and strain. His business is to know. If the bridge were to fall, he would go down with it in disgrace and ruin. Yet withal, he has never a fear. No mere experiment has he wrought, but a highway to carry men and women from shore to shore, to carry them secure and unafraid, though floods rage and boil below.

Well might we schoolmen voice the same lament, for whether the youth that we send forth from school will stand the stress and strain of life, go forward to the other shore secure and unafraid, few if any of us can say. Often we speak too glibly, I think, about our ability to know and fully develop the potentialities of the human individual by the time he is ready to leave school.

The graduate of the terminal course at secondary school at 18, and even the college graduate at 22 is not, strictly speaking, a finished product. He is, at most, or should be, a youth of great possibilities, or better still, probabilities, resulting from the strength of character, the right habits of mind and the knowledge we have given him. Whether these probabilities will be realized and what the finished product will be, we are as yet unable to say.

The peace of mind that comes to the designer of the bridge comes not to us for a number of years if it comes at all. In truth, we design too many bridges that fall, but when our work has been a failure, we do not know it until long after, so long after that we are not even held at fault. We are saved the disgrace and ruin as we are denied the peace of mind.

The best we school men can say even of those who appear to be the strongest among our graduates is that "the probabilities are" they will stand. We do not know, we cannot know. There is no inevitableness about it. We may speak only of probabilities.

How to make these probabilities strong, to wind cables of character that will not snap, lay habits of mind that will stay in place, and give knowledge that will not become obsolete, is the problem before us in considering modern preparation for college.

The public schools, of course, have to consider things other than preparation for college, because only about half of our secondary graduates intend to enter college. Under the law we must provide an education for all. We have not the selected college preparatory group that once we had and that the independent schools may now have. A large part of our secondary school program consists of vocational training for those who have no intention of going to college.

But the problem seems to me to be fundamentally the same whether we are preparing our youth for further study in an institution of higher learning, or for immediate entrance into the business of making a living. In either case, the probability that he will "stand through" to the end, "though floods rage and boil below," though changes more far-reaching than

* An address delivered upon the occasion of the Thirtieth Anniversary of the Riverdale Country School, at the Ritz-Carlton Hotel, Friday evening, March 19, 1937. Reprinted by permission from *High Points* for May, 1937.

any we have yet seen take place, must be as strong as we can make it, and it will be strong only if we have given him, in addition to strength of character and the ability to think straight, knowledge that is of real and permanent value. It must be an education of such content that its possessor may be of service to humanity irrespective of what technological, social or political changes may take place tomorrow or the next day. An education of any different content has a value subject both to depression and obsolescence.

The problem is chiefly one of what shall constitute preparation for a general college education. Preparation for special fields is determined largely by the specialty itself, and there is little difficulty in charting the course for one whose mind is made up to study medicine or law or prepare for the ministry. So, also, where one's talent is outstanding. The trial comes when the preparation must be general.

How to prepare the youth of no particular talent or aptitude, or whose mind is not fully made up, is that which causes our concern.

How must such a youth serve humanity and thus fulfill the purpose of his being? We, and often he, does not know, until after he has left us. His own wishes do not necessarily decide the matter. Circumstances may be the determining factor. We cannot foresee every eventuality, but we do know that the broader his training is, the more likely it will be that he may serve in one capacity or another. And the broader his training is, the more basic it must be. Here we are on solid ground, for when we deal with basic training we are dealing with definite subject matter and certain objectives. Indeed, we are right back where we were in the days before education reached the confused state of which President Hutchins complains.

The confusion came when pedagogy, affected by the demagogy of a demagogic age, veered to the left and urged that education should no longer consist of what the best judgment of the centuries had agreed to be a permanent value, but rather what modern youth in his infinite wisdom might think would be valuable. It came, too, when certain fields of education become commercialized and when the training of teachers became "big business." There had to be more students if there were to be more teachers to train, and more and different courses to give teachers when the old saturation points had been reached.

It took master teachers really to teach the traditional courses, and not all teaching training institutes were as much interested in training master teachers as in training more teachers. And the traditional courses, like everything else worth while, were difficult and required long hours of study. Easier courses would attract many more students.

So the traditional curriculum was declared outmoded. We must prepare, it was said, for a new social order, whatever that might mean.

Would not the laws of mathematics and physics remain in force? Would not history continue to have its significance, literature its power of inspiration? Would not Latin and Greek continue to be the source of the very language we speak, the basis of our art and culture and of our laws? Few seemed to think of this, in the rush for something new in the way of a general education. Knowledge, like wealth, was to be distributed more widely, albeit thinly.

Some called for subjects of a more practical nature, some for a "process of personality development," others for a curriculum of controversy. But above all, it must be something new and something different. The stock market had crashed, one party had replaced another in power at Washington, and a great change had come over the complexion of all things. Therefore, the content of a general education must change. It was all just about as logical as that.

Some have had sober second thoughts, realizing at length that no matter what our social or political philosophy may

be, civilization, as it grows more complex, will be more than ever dependent upon the men who know the laws of mathematics and the formulas of science, upon men who, having studied the history of mankind, know the relationship of the present to the past, upon men who can write and men who can speak the languages of other men. In truth, civilization will be more than ever dependent upon the very subjects that are included in the traditional curriculum for a general education.

One mistake of the past was that too many of our youth were led to believe that an education is an end in itself rather than a means by which one makes himself capable of being useful to humanity. To be really useful one must have definite and dependable knowledge of permanent rather than of transient value, and ability to apply the knowledge practically.

The subject of a general education should be definite and of such certainty that it may be taught as truth, lest men of doubt, distrust and wavering opinion go forth from our schools and colleges, educational agnostics, believing in nothing with no faith even in themselves.

If I have appeared to over-emphasize knowledge, it is because knowledge has been so slighted of late, so snubbed by a world in which everything that once was right is now considered wrong.

But if we give our youth understanding of the fundamentals in the major fields of human knowledge, cultivate their appreciation of art, literature and music, and if as well we make them thoughtful, instill in them a determination to do their work thoroughly and well, with the conviction that true happiness is the result of accomplishment and service, we give them things that are of real and enduring value, and the probability is strong that the bridge will stand from shore to shore.

Random Notes on Geometry Teaching

By HARRY C. BARBER

Note I
Geometry to Justify Itself

THE now-popular educational doctrine, —teach less mathematics; teach something that will make better citizens—is already diminishing the enrollment in secondary school geometry. Perhaps the enrollment in geometry ought to be reduced; perhaps, in this age, that would be a serious blunder; perhaps it makes little difference either way.

Ours is a scientific age; and a little consideration of the matter will convince anyone that we win our successes when we use the scientific method, when we adopt the scientific attitude of mind. The education of the intellect can have no more important aim than the development of this habit of mind called scientific. The fact finding, hypothesis forming, hypothesis testing, and unprejudiced reasoning which are characteristic of this habit of mind are exactly what we need for the solution of our problems of every sort.

Now how does elementary mathematics differ from other available secondary school subject matter?

First. It allows the learner to proceed under his own power, to reason his own way along from point to point. Learning something in mathematics is not usually memorizing it but discovering and understanding it. The successful pupil realizes this distinction, and will, if sufficiently articulate, speak about the difference between his study of mathematics and of his other subjects.

Second. Of all the fields of man's thought, mathematics is the one in which he has been most successful. Take politics, philosophy, language, economics, or what you will, none can compete with science in the completeness of its mastery of its field. And it is when science reaches the mathematical interpretation or the mathematical basis of its problems that success is most certain. (It is quite possible that this point may have entirely escaped the advocates of "less time for mathematics" because of the nature of their training.)

Third. Mathematics exhibits, as no other elementary subject can, just what a science is and how it proceeds. Under our own eyes its organization develops, without difficult or controversial material, into a science which is complete enough to show the essential nature of a science, and simple enough to be readily comprehended.

Finally. It is then evident that those things which make mathematics unique among high school subjects are precisely those things which enable the pupil to develop the scientific attitude of mind.

And furthermore the field of geometry is particularly rich in opportunities for learning to formulate definitions—which must be precise or else they impede the argument—for practice in precision of diction, for unprejudiced reasoning from known relations to wanted relations, and for the if-then type of thinking which makes directly and naturally for orderly habits of thought.

Mathematics and the social studies are, at the present moment, great rivals for the pupil's time—with mathematics on the losing end. Of course the social studies people are right about the great educational value of some of their subject matter, But its inherent weaknesses are just as obvious and, and they are precisely those suggested by the paragraphs above.

The social studies present to the pupils large bodies of facts to be memorized. They are constantly involved in matters of opinion; of racial, political, economic, or other prejudice and their pupils in high school are necessarily largely lacking in background, experience, and in the analytical scientific habit of mind which

are essential for the clarification of the issues involved.

Instead of being as contentious as they are just now, and as demanding of other people's time, the social studies advocates ought to demand that their courses be preceeded by and paralleled by other courses in which predjudice and emotion do not interfere with the development of logical habits of thought; courses in which the primary aim is the development of the scientific attitude of mind—that is, courses in mathematics.

One can be reasonably certain that future historians of American education in this era—education which is designed to prepare for life in a scientific age—,will find very amusing and very specious, to-day's plea for less mathematics. And we ourselves ought candidly to admit that these historians will put some of the blame upon those of us who are now teaching elementary mathematics, because we have permitted to become current so restricted a view of the place of mathematics in education.

A Billion?*

An after-dinner company were discussing Federal finances when one of the group observed, "Well, just how much is a billion, anyhow?"

"Why, a billion is a thousand millions," someone replied. "Or, if you like it better, it is a thousand thousand thousands."

None of us had ever seen a million of anything, much less a billion. Indeed as we discussed the matter, it seemed doubtful if any of us had ever actually *seen* as much as a thousand of anything tangible.

To be sure, a number of us had driven a hundred thousand miles or more in the course of years, and we had all seen assemblages said to number fifty, sixty, even eighty thousand or more. But had we seen them as separate individuals, or merely as a mass of humanity?

The more we discussed the matter, the clearer it became that the sense of number, in its higher values, is soon lost. Our impression is one of mass instead of individuals, and as we approach the million mark our comprehension is little more than a mental abstraction.

So it was that each of us took paper and pencil and set to work in an effort to expression a billion in terms sufficiently familiar to give something like an adequate conception of its meaning as a measure of values. The results of our efforts in various directions were substantially as follows:

It has been only a little over a billion minutes since Christ was born in Bethlehem.

If some man had spent a thousand dollars the day that Christ was born and every day thereafter as long as he lived, and if one of his descendants had continued to dispense funds at the same rate ever since, it would be some eight hundred years yet before a billion dollars would be spent—that is, around the year 2736.

Somebody announced that his calculations showed that it is only a little over a billion and a half inches around the world at the equator, and but slightly more than a billion and a quarter feet to the moon.

To change the picture, we found that a man whose pulse is around eighty to the minute will be almost twenty-four years old before his heartbeats amount to a billion.

It is a rapid speaker who can enunciate a hundred and fifty words to the minute. But if one could do that and hold out at the same rate for eight hours a day, month after month, indefinitely, it would take him almost thirty-eight years to deliver a billion words, with no time out for Sundays, holidays, or vacation.

A good ear of corn has about a thousand kernels. But it would require some twenty thousand bushels, or a matter of five hundred tons, of shelled corn to count out a billion kernels.

And if a man should set out to count the kernels in that twenty thousand bushels, and could maintain a rate of a hundred to the minute for eight hours a day, it would take him a matter of fifty-seven years to finish the job.

A forest with twenty-five mature trees to the acre is good lumbering. But to furnish a billion trees at the same rate would require an acreage equal to the combined area of New Hampshire, Vermont, Massachusetts, Connecticut, Rhode Island, New Jersey, Delaware, and most of Palestine.

A billion dollars is the total value to the farmer of the normal corn crop raised on an acreage equal to the combined area of Maine, New Hampshire, Vermont, Massachusetts, Connecticut Rhode Island, Pennsylvania, and most of New York.

A billion dollars would pay the farmer for all the cows and hogs of the United States, or for all the horses and sheep on American farms. It is nearly twice the farm value of all the beef cattle and calves before the day of the Great Slaughter.

In dollar bills laid end to end, one billion would reach one million, one hundred and sixty-seven thousand miles. This is more than forty-six times around the earth, and enough to reach to the moon and back *twice*, with some two hundred thousand miles to spare.

Finally, the total Federal debt amounts to one dollar for every hen's egg laid in the United States in one year, according to the reports of the United States Department of Agriculture.

* Reprinted by special permission from the Atlantic Monthly for November 1936, page 640.

Arithmetic Readiness and Curriculum Construction*

By Ben A. Sueltz
State Normal School, Cortland, N. Y.

It is desirable to indicate briefly what this topic implies. What do we mean by arithmetic, by arithmetic readiness, and by a curriculum?

A curriculum might be described as a desirable arrangement of selected materials or experiences for the guidance of pupils. Some person or group must make this selection lest like Topsy the curriculum "just grows." For this discussion it is assumed that there is value in planning in the light of experience.

Today the word arithmetic is just a bit puzzling. We are all familiar with the historical development of arithmetic as a body of knowledge and its prominence as a subject for study. In our own country the textbooks by such men as Daboll, Colburn, and Greenleaf tended to standardize not only the definition of arithmetic as "the science of numbers and the art of computation" but also these textbooks were the curriculum. More recently the rapid growth of what we naively call "scientific research in education" has further narrowed the common understanding of arithmetic by emphasizing abstract computations. Is it not true that although arithmetic may have benefited it has also suffered at the hands of neophyte researchers who often are unappreciative of its nature and scope?

During the past decade arithmetic (usually narrowly visioned) has been damned in many sections of the country. Even criminals have testified that their careers in crime had their origins in failure in arithmetic. The most drastic curriculum proposal has been to abolish the subject in the schools. Perhaps much of what we have been calling arithmetic should be abolished. Have our syllabi, our textbooks, workbooks, and our researches and investigations glorified the abstract science of numbers in such a way that children are often uninterested?

For the word *arithmetic,* I should like to substitute the word *mathematics* or the more descriptive term *mathematics for elementary schools* as we have done in New York State. The term mathematics has a broader connotation than arithmetic which to many people consists of a series of abstract processes. Instead of the "$2+2$, 7×8, and 6% of \$100" vision of arithmetic we might wisely choose one that features, concepts and ideas, principles and relationships, computations and manipulations, as these are really used in thinking and in arriving at conclusions in intelligent living. A curriculum that features all of these in terms of their social utility will include many geometric and algebraic ideas and principles. Functional thinking begins in the kindergarten as is evidenced by the expressions, "she is so slow it will take her a long time" and "I need a long thread because I have a lot to sew." Readiness to begin arithmetic depends upon what we mean by arithmetic. Let us take the broader interpretation which features concepts and principles as well as computations.

Arithmetic readiness involves the cooperation of all of those factors which combine to enable a child to explore and study mathematics profitably, We do not know much about this but we are reasonably sure that some of the factors involve:

a. sufficient physical and mental maturity to interpret an experience or to sense a situation.
b. an ability to discriminate size, quantity or magnitude.

It should be apparent that readiness for the kind of arithmetic or mathematics

* An address given at the Detroit meeting of the National Council of Teachers of Mathematics on June 29, 1937.

that I described is a growing or changing thing. For example there might be readiness for a concept of comparative size, or of shape, or of number and yet no readiness for the measurement of size, or the technical description of shape, or of computations with numbers. Mathematical readiness seems to depend upon the experience of the child, his maturity, and the nature of the mathematics.

Among young children one often finds rather remarkable mathematical insight. Consider the following in terms of potential curriculum experiences:

a. a 14-month old child arranges discs on a pin in order of size. Is this not an early concept of magnitude and order?
b. a 3-year old child arranges a small army of large toy soldiers against a larger army of small soldiers and trades two small for one large.
c. a 4-year old boy lays forks on the table and gets two more from the silver chest when exactly two more are needed.
d. a 7-year old girl was hit on the cheek by a swing and said "it's a good thing I wasn't a foot farther away or it would have hit me in the eye" (the rising arc of swing noted).
e. an 8-year old seeing two columns of blocks side-by-side and with one on top said there were 11, 13, 15 or some number like that. But when he noticed that the sizes were not uniform he said he would have to count them to be sure.

Each of you can match these experiences with others from the children in your own neighborhood. They show a readiness for concepts and relationships. However this is not yet a readiness for combinations like $5+9$ and $33 \div 4$ in written form. It is very difficult to write a syllabus or a textbook which features concepts, principles, and comparisons of size and order. Ideally these should come from the child's normal experiences in the school, the home, and the community.

The lack of sound concepts is apparent in every schoolroom where one hears such expressions as:

a. "I've got 56 fingers" (a 5-year old).
b. "Miss Smith is 100 years old, she has grey hair" (6-year old).
c. "There were a million people at the circus" (9-year old).

Ability to read and write numbers, an ability which schools have stressed, should not be confused with the ability to understand their value or magnitude. It is important to note that there is a great deal of mathematics which children learn and use before they need to learn to read and write. The possibilities of a curriculum that features informal, oral, and mental development of concepts and principles leading to number facts without the complicating use of paper and pencil should be further investigated.

Some informal work with five- and six-year old children leads me to believe that the focalization of number symbols is less confusing than that of letters and words. It seems reasonable to suppose (and it is only a supposition) that the reading and writing of numbers should come simultaneously with reading and writing words and sentences. However, since for many children the final abstraction of a number concept is a slow development, these concepts should be fairly well formed before the reading and writing stages. At present there seems to be much more artificial stimulation to read than there is for number concepts and principles. Some of us are beginning to feel that reading has been prematurely taught and that number ideas have been too long delayed.

Readiness for arithmetical processes and for problems that depend upon these processes in written form seems to be a different thing than readiness for the early stages of concepts and relationships. In this more complicated work several factors seem to interplay:

a. the interests and incentive of the child.
b. the essentially logical or sequential character of much of the work.

c. the mental maturation of the child. These factors must be treated together because they are cooperative for the learner. In a recent syllabus program we collected reports of children's experiences and interests in an attempt to fit the curriculum to the child. In the past, books have tended to emphasize the logical features of arithmetic. The "committee of Seven" has studied the mental age necessary for the satisfactory learning of processes. In a curriculum program these factors cannot be separated, they must be considered cooperative. The danger of isolating them is apparent in a widely quoted "research conclusion" (from p. 45, "Adjusting the School to the Child" by Washburne) which indicates that children "must have reached a mental age of 12 years and 7 months before they can do good work in long division." Weren't you tempted to smile when you first read and noted the specificity of the age 12 years and *7 months?* Because civilization seems to expect that schools should teach children to read and write and to able to do some figuring and since some children, who are apparently normal, would rather not learn to do all these things WHEN and HOW the teachers want them done we have had a great deal of piecemeal research seeking an easy way out. The "research conclusions" have not always agreed among themselves and they have often been at variance with the experience of hundreds of rather capable teachers. Is it wise to follow these researches or must we ask questions such as the following:

a. how many pupils were used in the study?
b. what type of teaching and learning procedures were used in previous work which may have been foundational to the present study?
c. what teaching and learning procedures were used in this study?
d. what types of tests and measuring instruments were used?
e. would you or anyone else in possession of all of the data draw the same conclusion?

Referring again to the "12 years and 7 months," if you had all of the information that I have indicated, I think you might be led to draw rather different conclusions and they might not be in terms of the mental age necessary to do long division.

At the present time we have no sound criteria for fitting mathematical processes to the age-grade level of pupils. The factor of interest and incentive is so unpredictable that we may have to conclude that the problem is highly psychological rather than purely mental or sociological. Perhaps the sane judgment of groups of teachers who approach their job from an inquiring and experimental point of view is still a better index of age-grade placement than our supposedly scientific investigations.

Let me conclude by summarizing.

1. The term mathematics for elementary schools may be well substituted for arithmetic because of the narrow connotation currently attached to arithmetic.

2. Arithmetic readiness is conceived as a growing or maturing readiness to meet different types and phases of mathematical experiences.

3. Young children often exhibit remarkable evidence of readiness for mathematical thinking which involves basic concepts and principles but not written computations.

4. Curriculum construction should capitalize the informal mathematical experiences of children through oral and mental work.

5. We know little about genuine readiness for arithmetical processes. The psychological element seems unpredictable. Researches and scientific investigations have tended to isolate factors that ought to cooperate.

Herbert Ellsworth Slaught

PROFESSOR HERBERT ELLSWORTH SLAUGHT, honorary president of the National Council of Teachers of Mathematics, passed away on May 21, 1937, at his home in Chicago in his seventy-sixth year. Professor Slaught's death removes from the scene of action one of the most devoted servants of the cause of mathematics in this country. For many years he was active not only in the affairs of the National Council, but also in those of the Mathematical Association of America, The American Mathematical Society, the Central Association of Science and Mathematics Teachers and several local organizations. He was a leader in the best sense. He was interested not only in promulgating and encouraging research activities, but was also active in stimulating others to study and improve the teaching of mathematics in secondary schools. He was instrumental in founding the Mathematical Association of America in 1916, and the National Council of Teachers of Mathematics in 1920. His death is a distinct loss to all the mathematical organizations that he served so long and so well.

Professor Slaught was born on a farm near Watkins, New York, on July 21, 1861. He was a self supporting student through high school and college at Hamilton, New York, graduating from Colgate Academy in 1879 and from Colgate University in 1883. He received the M.A. degree in 1886 and an Honorary Sc.D. degree from Colgate in 1911. He received the Ph.D. degree from the University of Chicago in 1898.

After teaching at Peddie Institute in Hightstown, New Jersey, for three years, he was promoted to assistant principal in 1886 and to the principalship in 1889. He accepted a fellowship in mathematics at the University of Chicago in 1892 when that University was founded. He subsequently served as reader, associate and assistant and instructor in mathematics from 1894 to 1897. He was made an Assistant professor in 1900, associate professor in 1908, professor in 1913, and professor emeritus in 1931.

Professor Slaught's great contribution was as a teacher and as an inspiration to his students, many of whom either were teachers or later became teachers. Many students who had studied calculus before, learned what the subject really meant in Professor Slaught's class. He was a kindly, intensely human, and understanding teacher who knew how to sympathize with ordinary students as well as to select those whom he wished to spur on to more advanced work. Many students of his will never forget the pleasant hours spent at his home with his wife and daughter Katherine at the teas on Sunday afternoons. His influence on the students at the University of Chicago both inside and outside of class was invaluable. We have all lost a helpful and true friend and the cause of mathematics has suffered an irreparable loss.

W. D. REEVE

THE ART OF TEACHING

A NEW DEPARTMENT

The Treatment of Algebra Problems

By C. Shanok

I believe that we may accept as axiomatic that one of the greatest sources of difficulty, if not the greatest, for the ordinary algebra student is the solution of verbal problems. It is my firm belief that this difficulty is very largely due—I am almost tempted to say solely due—to the method of attack commonly employed in illustrating the solution of such problems. If I make no mistake, this method of attack smacks considerably of that familiar fallacy, putting the cart before the horse. Bold words, perhaps, but I hope that you too will concur in this sentiment after I have completed my argument.

Let us proceed now to point out wherein the alleged error lies. For this purpose, let us consider the following problem: "Two automobiles starting at the same time, traveled a distance of 60 mi., one going six miles per hour faster and completing the journey five-sixths of an hour sooner. How fast did each automobile travel?" Starting in the customary manner, we let x represent the number of miles per hour of the slower car. It then follows that $x+6=$ the number of miles per hour of the faster car. We now see that $\frac{60}{x}=$ the number of hours required by the slower car and that $\frac{60}{x+6}=$ the number of hours required by the faster car. We now proceed to write our equation in the following manner: $\frac{60}{x+6}=\frac{60}{x}-\frac{5}{6}$, because the faster car took five-sixths of an hour less time. Solving, we find that $x=18$ which is the rate in miles per hour of the slower car, and that $x+6=24$, which is the rate in miles per hour of the faster car, and our problem is completed. This is all very lovely, and we feel an inward glow of satisfaction over the thought that we have taught our pupils how to do this problem to their complete satisfaction. But have we? Suppose for instance, that one of the pupils asks why we had to express the *times* in terms of x. "Why," he asks, "don't we express the *distances* in terms of x, as we did in the time, rate, distance problems we have already learned to solve?" We might perhaps answer that we don't express the distances in terms of x because in this problem the distances are known. But more mature reflection will convince us, I feel sure, that we have not found the correct answer to this question. "What, then, is the answer?" you ask. The answer, as some of you have perhaps already realized, is that the equation statement, in our problem the statement that one car completes the journey five-sixths of an hour sooner than the other, deals here with *times*, and therefore it is necessary to express the *times*, rather than the *distances*, in terms of x. This point established, I believe you will now readily agree that there is a flaw in the above solution, namely, that the pupils are told to express the *times* in terms of x without the reason for this step being given. How shall we correct this flaw? The answer—if I make no mistake—is to insert the following explanatory step into our solution: (the translating of the equation statement into an equation in words)

"time of faster car = time of slower car $-\frac{5}{6}$." As to the question "Where shall we write this step?" there seems to be little room for argument. This explanatory step should be the first step of our solution, because it is this step which explains why we must express the times in terms of x.

We are now ready to outline the steps of the solution in their correct order.

Step 1. Find the equation statement and translate it into an equation in words.

Step 2. Express each of the quantities called for in the equation in words in terms of the letter x. (We note in passing that the choice of x is part of this step.)

Step 3. Substitute for each of the quantities called for in the equation in words its expression in terms of x and solve the resulting equation.

Step 4. Answer the questions called for in the problem.

Perhaps it will be well at this point to illustrate the method by another problem." A photographer has two bottles of diluted developer. One is a 10% solution and the other a 50% solution. How much must he add from the first bottle to 8 oz. from the second bottle to make a 20% solution?" Our equation in words here is that the amount of developer in the first mixture + the amount of developer in the second mixture = the amount of developer in the final mixture. We have now that $0.1x$ = the amount of developer in the first mixture, where x is the amount taken from the first bottle, that $0.5(8)$ or 4 = the amount of developer in the second mixture, and that $0.2\ (x+8)$ = the amount of developer in the final mixture. Thus we get the following equation: $0.1x+4=0.2\ (x+8)$. Solving, we find that $x=24$ ounces which is the amount taken from the first bottle.

In conclusion, there are just two points I would like to make. In the first place, mention should be made of the fact that the credit for this method is due to Frank C. Touton, late Vice-President of the University of Southern California. The paper in which he dealt with this topic is entitled "The frequency of certain problem solving situations in the high school curriculum and a suggested general method of solution." This paper appeared in the April 1922 number of "School Science and Mathematics." Credit is also due to Barnet Rudman for his article in the February 1929 number of the Mathematics Teacher entitled "Teaching the Verbal Problem in Intermediate Algebra."

In the second place, I don't want to leave the impression that the method outlined furnishes a royal road to the solution of problems. Even granting that this method is the correct approach, there is still a great amount of work to be done, especially in building up new lists of problems, so arranged as to facilitate the teaching of this method.

◆ IN OTHER PERIODICALS ◆

By Nathan Lazar
Alexander Hamilton High School, Brooklyn, New York

1. Aitken, A. C. *Trial and Error and Approximation in Arithmetic.* The Mathematical Gazette. 21: 117–122. May 1937.

A "mathematical wizard" discusses some simple methods of approximation, as well as the nature of arithmetical memory and faculty.

2. Devisme, Jacques. *English and French Teaching Methods Compared.* The Mathematical Gazette. 21: 123–134. May 1937.

A French professor of mathematics compares the ideals and the teaching methods of the French and the British by a careful, analytic study of the contents in representative textbooks of the two countries.

3. Forbes, Raymond F. *Mathematics Functioning in Industry.* School Science and Mathematics. 37: 513–519. May 1937.

Interesting comments by the head of the research department of the Indianapolis Power and Light Company on the importance of mathematics for those engaged in various phases of industrial work.

4. Gabriel, R. M. *The History of Mathematics: Its Relation to Pupil and Teacher.* The Mathematical Gazette. 21: 106–112. May 1937.

Many places are pointed out in the teaching of mathematics where the introduction of appropriate historical material would make the topic taught more meaningful and more interesting.

5. Hardy, G. H. *The Indian Mathematician Ramanujan.* The American Mathematical Monthly. 44: 137–155. March 1937.

A biographical sketch of "the most romantic figure in the recent history of mathematics. . . He was at best, a half-educated Indian; he never had the advantages, such as they are, of an orthodox Indian training. . . . He worked, for most of his life, in practically complete ignorance of modern European mathematics, and died when he was a little over 30 and when his mathematical education had in some ways hardly begun."

6. McKee, Charles Lester. *Historical Material in Secondary Mathematics.* School Science and Mathematics. 37: 588–579. May 1937.

The author reports the results of an investigation he conducted in an attempt to find the answers to the following questions:

(1) "What is the degree of consistency in the content and amount of historical material presented in secondary mathematics texts?

(2) "Just what is the value of historical material in secondary mathematics?

(3) "Do high school texts deal with this material in an adequate way?"

A good bibliography of the relevant literature is included.

7. Porges, Arthur. *The Application of Continued Fractions to the Determination of Empirical Chemical Formulas.* School Science and Mathematics. 37: 598–599. May 1937.

In this article it is pointed out how a knowledge of continued fractions enables the chemist to obtain the formula of a compound like $Mo_{24}O_{37}$,—"a result impossible to obtain with the ordinary methods at the disposal of the student of theoretical chemistry."

8. Punnett, Margaret. *Mathematical Films.* The Mathematical Gazette. 21:149–151. May 1937.

A report, on behalf of the Film Sub-Committee, that deals with films illustrating various mathematical formulae, and theorems, and mechanical laws and motions. A list is included that contains the names and addresses of producers, prices, etc.

9. Read, Cecil B. *Mathematical Magic.* School Science and Mathematics. (a) 37: 597. May 1937. (b) 37: 650. June 1937.

A few tricks with numbers depending on the fact that the remainder obtained by dividing a number by nine is the same as the remainder found by dividing the sum of the digits by nine.

10. Stafford, Anna A. *Adapting the Curriculum to Our Era.* School Science and Mathematics. 37: 400–415. April 1937.

The writer concludes her appeal to the teachers of mathematics with the following remarks:

1. "Let us keep ourselves steeped, as far as in us lies, in what really goes on in mathematics.
2. "Let us boldly fall in with the people who want to discard parts of the mathematics curriculum whose value is overemphasized, purely historical, or in any case dubious.
3. "Let us not fear to make innovations whose social value is obvious. I cannot see that that is a disgraceful surrender of conservative mathematics.
4. "Let the new textbooks be written with the teacher in mind. . . . "

A bibliography is included.

11. Schuler, Eucebia. *The Professional Treatment of Freshman Mathematics in Teachers Colleges.* School Science and Mathematics. (Part I) 37: 464–472. April 1937. (Part II) *Application of Professional Treatment to the Quadratic Function.* 37: 536–548. May 1937.

In Part I the author enumerates the main points which should be emphasized in professional content courses in any subject matter; in Part II the general scheme is worked out in great detail in connection with the quadratic function.

An extensive bibliography is included.

12. Tracey, J. I. *Undergraduate Instruction in Mathematics.* The American Mathematical Monthly. 44: 284–288. May 1937.

Many interesting comments on an important topic. "An improvement in undergraduate instruction including more real teaching, and perhaps less talking on part of the instructor, will turn better teachers of mathematics into the secondary schools in a few years; and if these in turn emphasize the same quality and thoroughness of instruction we may eventually succeed in getting arithmetic taught properly in the elementary schools."

13. Wolff, Georg. *The Development of the Teaching of Geometry in Germany.* The Mathematical Gazette. 21: 82–98. May 1937.

After a detailed exposition of the teaching of geometry in Germany and in England, the author concludes "that in these two countries above all we have the common tendency of development: *Away from Euclid.*"

NEWS NOTES

The Joint Commission on the Place of Mathematics in the Secondary Schools plans to issue a preliminary partial report early in 1938. Composed of members of both the Mathematical Association of America and the National Council of Teachers of Mathematics, the Commission has been considering its problems since 1935. A grant from the General Education Board received in January, 1937, made possible a series of meetings which have led to the forthcoming preliminary report.

The Commission believes that judgments concerning the place of mathematics in the schools should be reached by a careful consideration of general educational aims. Accordingly the report will include a discussion of relevant educational problems, followed by a set of principles useful in the construction of a broadly conceived program of general education. The report will also discuss some of the objectives of secondary education, such as the capacity to think clearly, the ability to use concepts, information, general principles, and skills, and the development of attitudes, interests, and appreciations. The contribution that mathematics makes towards the achievement of such objectives on the part of the individual, as well as the role of the subject in civilization will be considered.

After presenting an outline of the general content of the main divisions of elementary mathematics, the report will offer criteria for the selection and organization of materials, and appraisal of pupil progress. In view of present diversity of administrative organization and pupil needs, the Commission believes it desirable to avoid narrow and rigid prescriptions. It will, however, offer some suggestions for programs for grades seven, eight and nine, for grades ten, eleven and twelve, and for the junior college.

The Commission has still to consider modifications for the suggested program, problems centering around the slow-moving and around the superior pupil, as well as the question of teacher training. It hopes that suggestions growing out of the study of the preliminary report will be helpful in its further deliberations.

Members of the Commission are as follows: *Representing the Association*—K. P. Williams, Chairman, Indiana University, Bloomington, Indiana; A. A. Bennett, Brown University, Providence, R. I.; H. E. Buchanan, Tulane University, New Orleans, Louisiana; F. L. Griffin, Reed College, Portland, Oregon; C. A. Hutchinson, University of Colorado, Boulder, Colorado; H. F. MacNeish, Brooklyn College, Brooklyn, New York; U. G. Mitchell, University of Kansas, Lawrence, Kansas.
Representing the Council—William Betz, Rochester Public Schools, Rochester, New York; M. L. Hartung, University of Wisconsin, Madison, Wisconsin; G. H. Jamison, State Teachers College, Kirksville, Missouri; Ruth Lane, University High School, Iowa City, Iowa; J. A. Nyberg, Hyde Park High School, Chicago, Illinois; Mary A. Potter, Supervisor of Mathematics, Racine, Wisconsin; W. D. Reeve, Teachers College, Columbia University, New York City.

The next meeting of the Commission will be held at Indianapolis, Indiana, during the Christmas holidays in connection with the meeting of the Mathematical Association of America.

It is probable that one session of the National Council of Teachers of Mathematics at Atlantic City on February 25th and 26th will be given to a discussion of the preliminary report. In the meantime and subsequently it is hoped that teachers of mathematics all over the country will find time to discuss the preliminary report which will be available later. For further details, watch the news notes in *The Mathematics Teacher*.

Many members of the National Council of Teachers of Mathematics will be grieved to hear that Miss Mabel Winspear, the most efficient Secretary of *The Mathematics Teacher*, passed away on September 4, 1937. Miss Winspear was recovering from an appendix operation at St. Luke's Hospital in New York City when a blood clot from the operation went to her heart and although she had the best of medical service, she passed away.

Miss Winspear served *The Mathematics Teacher* for many years. She was devoted to her work, was cordial and tactful in all her contacts with subscribers and in fact had a host of friends among the members of the National Council whom she met both in her office and at various Council meetings.

NEW BOOKS

Approximate Computation. By Aaron Bakst. Bureau of Publications, Teachers College, Columbia University, 1937. xvi+287 p. $1.75.

This interesting book serves or has served three purposes: (1) It is the twelfth yearbook of the National Council of Teachers of Mathematics, (2) It has been accepted as the author's dissertation for the degree of Doctor of Philosophy in Columbia University, and (3) It furnishes the material necessary for teachers who need to be better acquainted with the important subject of approximate computation, that phase of numerical work which enters so extensively into modern life. A fourth purpose, and by no means a minor one, may be found in the pleasure which it is destined to give to all who seek to know what a wide range of mathematics is included in the field of approximate computation in the divers walks of life. Furthermore it is interesting to come across a doctor's dissertation which seems really worth the effort to read, and of a candidate who undertakes to construct such a book. A generation ago a work of this kind would rarely have been looked upon as necessary in the training of teachers in general, or even those who were concerned with mathematics alone. Even now there is a tendency to look upon the colleges for the training of teachers as merely places for learning some of the elements of the theory of education with almost no attention to the necessity for absorbing more than a minimum of knowledge of the subject to be taught. Of course it may be said that we no longer teach "subjects," and all that is needed for a teacher to know about approximate computation is what is found in a mythical tour with his pupils along the Onondaga Indian Trail or a visit to Niagara Falls—an elaboration of the decadent 'project method.' For those who preach this doctrine—although it is said that it is not on the increase—books like this are merely "caviar to the general," hardly food to be assimilated by all.

Dr. Bakst begins by showing from original sources, such as reports of committees, textbooks, and bulletins issued by state institutions in various countries, that whereas the subject of approximations is of great importance, and is talked about (generally to little or no purpose), it is almost everywhere neglected more than any other subject of equal importance to teachers. He then quotes from the Report of the National Committee on Mathematical Requirements, showing that in algebra, geometry and trigonometry, whenever formulas are used the pupils should be able to differentiate between 'exact mathematical formulas' and 'approximate mathematical formulas,' and also between either and 'scientific formulas.' All this sounds well, but Dr. Bakst, not content to stop at mere sound, proceeds to show that the mere talk has had but little effect upon recent theories of education. Very wisely he then turns to the story of measurement, and in a few pages he furnishes the material for interesting events in the birth and growth of the various measures familiar to teachers and pupils. This leads him to the fascinating topic of indirect measurements, and of the errors which here arise as well as in the direct finding of dimensions.

The real test of such a book is not, however, found in what it might possibly accomplish in the abstract equipment of a teacher, but in supplying an adequate number of genuine applications for the teacher's use. It is in this field that Dr. Bakst has shown his real ability. To attempt to enter into details as to the wide range of applications which he provides would not be feasible in a sketch so limited as this. A few major headings must therefore suffice. These include the meaning of the term 'measure,' generally so loosely used; 'relative error,' 'approximations' in simple numerical work and in logarithms, the notion of 'optimum approximativeness' (one of those unnecessary terms, like 'pedagogy,' born by mistake of 'pedagogical' parentage, and then thrust upon authors), and approximations in algebraic work.

Having treated in an elementary way upwards of a hundred pages devoted to these topics, Dr. Bakst enters upon the subject of numerical processes, dwelling at length upon the more scientific treatment of the subject of approximations. This, the Part II of his thesis, furnishes a more thorough discussion of his problem—one more suited to college classes. In general the topics are similar to those in Part I, but they are viewed from a higher standpoint. Here the author is to be congratulated upon his restrained use of symbols, which is not so evident in many of our modern works of an algebraic nature.

In the closing pages Dr. Bakst gives a satisfactory bibliography, including the standard works on the subject in various European languages. In this he gives evidence of the remarkable progress in the subject made in Russia since the Revolution. Himself familiar with the major

language of the Soviet Republic, he has been able to make known the recent progress of that country in this rather abstract part of mathematics.

The book should find place in all college and university libraries, in those of the best grade of high schools, and in those of technical schools in which precision and accuracy of measurement are of maximum importance. It should also be in the personal libraries of all who wish to keep abreast with the advance in present-day acquisition of knowledge—which is not necessarily the same as 'going to school.'

Dr. Bakst is a careful writer, so presumably all his mathematical statements, formulas, and computations are as free from error as can be expected in the first printing of any mathematical text. Perhaps the absence of an index may be excused by the fact that there is an unusually complete table of contents.

David Eugene Smith

Numbers and Numerals. A Story Book for Young and Old, David Eugene Smith and Jekuthiel Ginsburg. *Contributions of Mathematics to Civilization*, Monograph Number One, Edited by W. D. Reeve, Bureau of Publications, Teachers College, Columbia University, New York City, 1937, viii+52 pp. Price 25¢ postpaid.

This monograph has been eagerly awaited since it was first announced some months ago. It is to be followed by "Great Men of Mathematics," "The Story of Measurement" and others whose titles are not yet listed. These will receive a cordial welcome from the friends of the first brochure in the series.

Louis Agassiz is said to have characterized a scholar as a man who can discuss his subject in technical terms with experts, who can express it in language that can be understood by the man in the street, or who can tell it as a fairy tale for children. People familiar with Professor Smith's works will classify his *History of Mathematics* and his *Rara Arithmetica* under the first heading and his *Number Stories of Long Ago* under the third. Numbers and Numerals, written in collaboration with Professor Ginsburg, belongs to the second class although the editor's note suggests that it may be used as supplementary reading material in school classes in mathematics and in the social studies. The more able pupil in the junior high school and in the senior high school presents much the same problem as does the average man of Agassiz's statement. It is from the point of view of using this booklet for supplementary material in schools that the following digest and comment has been written.

First, although some of the material and certain of the illustrations were used in *Number Stories of Long Ago*, one should not surmise that this is simply a reprint of the earlier book with the color plates omitted. The *Number Stories* covers a more restricted field at somewhat greater length, but the reader will have no feeling that *Numbers and Numerals* has suffered by being in a shorter form. While many of the cuts are identical, several important new ones have been added as for instance an Egyptian bas relief on page 11; a Roman inscription, page 15; and Gothic numerals, page 17.

The booklet treats number names, number symbols, computation and number recreations. The last chapter gives the history of certain words in the terminology of arithmetic.

On first reading, one is puzzled to know how to appraise this booklet properly for its fine points are many. Yet a review which suggested that the booklet were perfect would by its fulsomeness defeat its purpose. Accordingly, I shall strain at the gnats.

The preface promises a story of numbers, how they came into use, and how our number forms came to have their present shape. It speaks of "the first crude numerals, or number symbols." The preface has the statement, often novel to teachers, that "the idea of number separated itself from the objects to be counted and thus became an abstract idea. We shall also see how this abstract idea became more and more real as it came into contact with the needs of everyday life and with the superstitions of the people."

A gnat is found in the preface when the readers are told that they need not pronounce the foreign words in the text. It would have been difficult to work out a set of diacritical marks, yet they would have helped greatly. How else can you discuss the material,—how else can you give an oral report of your readings? Not many suggestions about pronunciation would have been needed.

Chapter I, Learning to Count, shows the necessary of counting in some way as a help in barter. The keynote of the chapter is the statement that "All day long we either use numbers ourselves, or we use things that other people have made using numbers." Number systems of certain primitive tribes are mentioned, and number names in French and in English are listed which show traces of 20 and of 12 as a base in counting.

Chapter II, Naming the Numbers, begins with a conjecture stated categorically, "Number names were among the first words used when people began to talk." This chapter contains a table of number names from one to ten in English, French, German, Old English, Latin and Greek. The high point in this chapter is the

statement in regard to the fact that in many languages, the names for one and first; two and second, bear no close relation to each other,—"All this goes to show that early people did not connect 'the second boy' with 'two boys.' "

Chapter III, From Numbers to Numerals, includes the numerals of many countries culminating with the Hindu-Arabic numerals, with suggestions about computation and a section on finger notation—a great deal to compress into seventeen pages. The illustrations are excellent, particularly those showing Egyptian and Roman numerals. The use of both 60 and 10 as a base in writing Babylonian numerals is hinted but not explained.

A dilemma appears in the table showing the twenty-seven characters of another set of Greek numerals. The exigencies of an alphabet of only twenty-four characters and the demands of a numeral system requiring twenty-seven, led the Greeks to use an obsolete Greek letter not in the classic alphabet and to borrow two others from another alphabet. This is not mentioned, but the authors state that "Q and Z are used in place of two ancient Greek letters not in our alphabet." The reader, ignorant of the Greek alphabet, will find many others, gamma, delta, theta, etc., that are not in our alphabet and unfortunately he will find Z in two places, for 7 and for 900 and this may puzzle him. On the other hand, the reader who remembers the Greek alphabet will find *F* after epsilon where he expects zeta. But *F* is not in the Greek alphabet unless he recollects the old digamma. It is really unfortunate that the exigencies of the type setting did not permit the use of the proper characters for 90 and 900.

Roman numerals are interestingly explained, especially in showing the wide variety of their use. From the point of view of a seventh or eighth grade pupil, however, there is another gnat in the discussion of Humphrey Baker's four CliM, two C xxxiiii, millions, sixe ClxxviiiM, five Clxvii. This is explained by the use of brackets and parentheses in a way probably unintelligible to the younger reader, while writing it as four hundred fifty-one thousand, two hundred thirty four millions; six hundred seventy eight thousand, five hundred sixty seven would have been clear when coupled with the note about the fact that what we call a billion was customarily called a thousand million in England in Baker's time.

The impression many people have that Roman numerals are difficult to use in computation is dispelled by specific examples in which they are used in addition and subtraction. This is a point which should be emphasized.

The changes in the form of Hindu-Arabic numerals in the past seven hundred years will be astonishing to those who suppose that these have always been as they now appear.

Chapter IV, From Numerals to Computation, includes the various forms of the abacus, schemes for multiplication and division with Hindu-Arabic numerals, and the use of knots both in the Peruvian quipu and in an instance in Germany, the approximate date of the latter being omitted. A facsimile is given of a page of Robert Record's (sic) *Ground of Artes* to show reckoning with ounters. It is too bad that more space was not devoted to explaining the form of the multiplication examples from the Treviso Arithmetic, and to discussing the scratch method of division. Few students can follow these without greater assistance than is given here.

It would be helpful if the picture from the *Margarita Philosophica* showing Typvs Arithmetiae had had a caption explaining how *Boethius* came to sponsor the computation with ciphers and *Pythagoras* the older computation with counters.

Fractions come in Chapter V. The term "measurement fractions" for the smaller units in denominate numbers is an expression that should be remembered. It would perhaps have been well to point out the difference between the raised dot used by the English and the period used by Americans as a decimal point, otherwise the distinction between 2·75 and 2.75 will not be noticed by many readers. The facsimile from Stevin's work on decimals might well have carried an explanation of his notation which would otherwise be somewhat obscure.

In discussing the dollar sign, the authors mention that "it came into use soon after 1800, but it was not generally recognized until about 1825 or 1830." Here the readers will wonder what *was* used before 1825. Dollars were being coined and dollars were being spent. How did people write them? This is a minor point, but our younger readers will raise it.

The chapters on the Mystery of Numbers and on Number Pleasantries will fascinate many readers. The statement regarding the size of the largest number that can be written with four figures is breath-taking. These chapters may answer the perennial question, "What is the biggest number?" This, as we know, is a thing that worries the more able of our junior high school students.

It will be noticed that the gnats that have been mentioned are for the most part things that affect the more immature or the more careless readers. It may be that the omissions noted have been made purposely to tease these people into activity in figuring things out for themselves and that the authors are sympathetic with Descartes who excused his hiatuses by the statement that he "left to others the pleasure of discovery."

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